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Degenerate and star colorings of graphs on surfaces

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ABSTRACT

We study the degenerate, the star and the degenerate star chromatic numbers and their relation to the genus of graphs. As a tool we prove the following strengthening of a result of Fertin et al. (2004) [8]: If G is a graph of maximum degree Δ, then G admits a degenerate star coloring using $O(\Delta^{3/2})$ colors. We use this result to prove that every graph of genus g admits a degenerate star coloring with $O(g^{3/5})$ colors. It is also shown that these results are sharp up to a logarithmic factor.

1. Concepts

Let $G = (V, E)$ be a graph. An *n-coloring* of G is a function $f : V \to \mathbb{N}$ such that $|f(V)| \leq n$. We say that f is a *proper* coloring if $f(x) \neq f(y)$ for every edge $xy \in E$. A *color class* C_i of f is the set $f^{-1}(i)$, where $i \in f(V)$. Two colorings f and g of G are said to be *equivalent* if the partitions of V into color classes of f and g are equal. Suppose that for each vertex $v \in V(G)$ we assign a *list* $L(v) \subset \mathbb{N}$ of *admissible colors* which can be used to color the vertex v. A *list coloring* of G is a coloring such that $f(v) \in L(v)$ for each $v \in V$. If for any choice of lists $L(v)$, $v \in V$, such that $|L(v)| \geq k$, there exists a proper list coloring of G, then we say that G is *k-choosable*. The *list chromatic number* of G, denoted as $\mathrm{ch}(G)$, is the least k, such that G is k-choosable.

A proper coloring of G, such that the union of any two color classes induces a forest, is called an *acyclic coloring*. The *acyclic chromatic number* of G, denoted as $\chi_a(G)$, is the least n such that G admits an acyclic n-coloring.

The notion of a degenerate coloring is a strengthening of the notion of an acyclic coloring. A graph G is *k-degenerate* if every subgraph of G has a vertex of degree less than k. A coloring of a graph such

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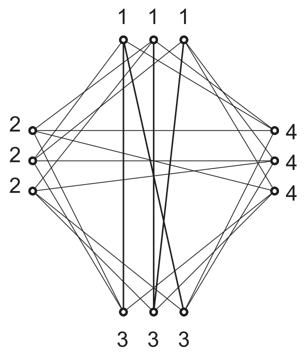

Fig. 1. An example of a star coloring which is not degenerate.

that for every $k \geq 1$, the union of any k color classes induces a k-degenerate subgraph is a *degenerate coloring*. The *degenerate chromatic number* of G, denoted as $\chi_d(G)$, is the least n such that G admits a degenerate n-coloring.

A proper coloring of G, with no two-colored P_4 is called a *star coloring*. This is equivalent to saying that the union of any two color classes induces a star forest, i.e. a subgraph whose every component is a star $K_{1,t}$ for some $t \geq 0$. The least n such that G admits a star coloring with n colors is called the *star chromatic number* of G, denoted as $\chi_s(G)$.

If a coloring is both degenerate and star, then we speak of a *degenerate star coloring*. The corresponding chromatic number is denoted as χ_{sd}.

A proper list coloring is an acyclic coloring if the union of any two color classes induces a forest. The *acyclic list chromatic number* ch_a is the least n, such that for any assignment of lists of size n, there is an acyclic list coloring of G. The definitions of list versions for all other types of chromatic numbers are analogous to their non-list versions and we denote the list versions of chromatic numbers by ch_a, ch_d, ch_s and ch_{sd}.

Clearly, $\chi_a(G) \leq \chi_d(G) \leq \chi_{sd}(G)$ and $\chi_a(G) \leq \chi_s(G) \leq \chi_{sd}(G)$. However $\chi_d(G)$ and $\chi_s(G)$ are not comparable. To see this, note that the degenerate chromatic number of a tree is two. However, for any tree T which is not a star, $\chi_s(T) \geq 3$. In Fig. 1 we give an example of a graph whose star chromatic number is four, but has no degenerate four-coloring (since its minimum degree is four).

It is well known that the list chromatic number of a graph of genus g is $O(g^{1/2})$ (see e.g., [11]). For acyclic colorings, Borodin proved in [6,7] that every planar graph admits an acyclic 5-coloring and thereby solved a conjecture proposed by Grünbaum [9]. Alon et al. [3] determine the (asymptotic) dependence on the acyclic chromatic number for graphs of genus g, where g is large. The corresponding bounds for the acyclic list chromatic number have not appeared in the literature, but the proof in [3] can be rather easily adapted to give the same bounds for the list chromatic version.

It is also conjectured in [5,7] that every planar graph can be colored with five colors, so that the union of any k-color classes induces a k-degenerate graph for $k = 1, \ldots, 4$. Rautenbach [16] proved the existence of degenerate colorings of planar graphs using eighteen colors. This result was recently improved to nine colors in [10]. However, for nonplanar graphs, this type of coloring has not been treated before.

In [1] it was proved that every planar graph admits a star coloring with twenty colors and that the star chromatic number of a graph of genus g is $O(g)$.

The aim of this paper is to establish upper and lower bounds for the degenerate and the star list chromatic numbers. We prove that the degenerate star choice number of a graph of genus g is $O(g^{3/5})$ thereby improving the bound $O(g)$ given in [1]. We also prove that our bound is sharp up to a logarithmic factor. These results in particular solve Problem 3 proposed in [1, Section 8]. The results of this paper and previously known results are collected in Table 1.

Nešetřil and Ossona de Mendez studied some of these questions in a greater generality. In particular, they considered star colorings in arbitrary minor closed families of graphs [13,14]. Cf. also [15].