European Journal of Combinatorics 33 (2012) 380–396

Contents lists available at SciVerse ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Maximum size of a planar graph with given degree and even diameter

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ARTICLE INFO

Article history:
Available online 23 November 2011

ABSTRACT

We offer the exact solution of the degree–diameter problem for planar graphs in the case of even diameter $2d$ and large degree $\Delta \geq 6(12d+1)$. New graph examples are constructed improving the lower bounds for $\Delta \geq 5$.

1. Introduction

We consider the degree–diameter problem restricted to planar graphs. We look for the largest number of vertices $p(\Delta, D)$ in a planar graph with maximum degree Δ and even diameter $D = 2d$. Hell and Seyffarth [3] have computed $p(\Delta, 2) = [3\Delta/2] + 1$ and proved that this value is exact for $\Delta \geq 8$. Fellows, Hell, and Seyffarth have also found [1] rather rough upper bounds $p(\Delta, 2d) = (12d+3)(2\Delta^d + 1)$ for $d > 1$, $\Delta \geq 4$. To this end they have applied the Lipton and Tarjan separator theorem [4]. Later [2], they have constructed plane graphs proving the lower bound

$$p(\Delta, 2d) = \frac{(3\Delta - 4)\Delta(\Delta-1)^{d-1} - 4}{2(\Delta - 2)}.$$

At the same time they emphasized "that the lower bounds are likely to be closer to the actual values of $p(\Delta, D)$ and that good upper bounds likely to be difficult to establish". They asked as well the question: "Let D be fixed. Is it the case that for all sufficiently large Δ there are networks with maximum degree Δ, diameter at most D, and $p(\Delta, D)$ nodes which are all of the same type?" We improve the constructions of Hell and Seyffarth increasing in the case $\Delta \geq 5$ the lower bound:

Theorem 1.1. *The maximum size of a planar graph G of diameter $D = 2d$ is at least*

$$p\,(\Delta, 2d) = \left[\frac{3\Delta}{2}\frac{(\Delta-1)^d - 1}{\Delta - 2}\right] + 1. \tag{1}$$

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0195-6698/$ – see front matter
doi:10.1016/j.ejc.2011.09.005

We show that this bound is exact for large Δ:

Theorem 1.2. *The size of a planar graph G of diameter* $D = 2d$ *and maximum degree* $\Delta \geq 6(12d+1)$ *is at most*

$$\left\lceil \frac{3\Delta}{2} \frac{(\Delta-1)^d - 1}{\Delta - 2} \right\rceil + 1.$$

The proof of Theorem 1.2 is based on a 5-separator construction in a plane graph. The existence of an N-separator in a plane graph with bounded number of vertices in each face was proved recently [5]. We use hereunder Theorem 1.1 of [5] in the case $N = 5$:

Theorem 1.3. *Given a plane triangular graph* G_T *and spanning tree* G_S *in it. If*

$$|V(G_T)| > 7,$$

then, there exists 5-separator S_5 *partitioning graph* G_T *into five parts* $A_i, i = 1, \dots, 5$ *with borders* $b_i, i = 1, \dots, 5$ *such that*

$$V(A_i) \geq \frac{|V(G_T)| + 2}{9} - \frac{V(b_i)}{2}. \quad (2)$$

2. Some preliminary results

We consider hereunder a plane graph G of maximum degree Δ, and diameter $2d$.

Claim 2.1. *The maximum number vertices at distance n from any given vertex* $v_C \in V(G)$ *is*

$$|V_n| \leq \deg(v_C)(\Delta - 1)^{n-1}. \quad (3)$$

Claim 2.2. *Let* R_n *be a root tree of length n in G and* v_R *be its root vertex. Then*

$$|R_n| \leq 1 + \deg(v_R)\frac{(\Delta-1)^n - 1}{\Delta - 2}. \quad (4)$$

Consider arbitrarily cycle C in G and its interior A_C. Let $V_n(A_C)$ be set of vertices of A_C at distance n from C. The following lemma holds.

Lemma 2.3. *Given three distinct arcs* $C_i, i = 1, 2, 3$ *of cycle C, and set* $V_{n3} \subset V_n(A_C)$ *any vertex of which is connected by n-paths with each part* C_i*. Then,*

$$|V_{n3}| \leq 3(\Delta - 1)^{n-1} - 2. \quad (5)$$

Proof. The proof is direct. Consider arbitrarily vertex v_C and three non intersecting (but possibly having common parts) n-paths $P_i, i = 1, 2, 3$ connecting it to three distinct arcs $C_i, i = 1, 2, 3$. These paths partition the interior of cycle C into at least three separate faces $F_i, i = 1, 2, 3$ (Fig. 1). Let $V_n(F_i) = V_n(A_C) \cap V(F_i)$. Consider arbitrarily face F_1. Since it has no vertices incident to vertices of $V(C_1)$ each vertex of $V_n(F_1)$ is connected to arc C_1 by an n-path which crosses the border of F_1 at some vertex $v_P \in V(P_2 \cup P_3)$. Without loss of generality, let $v_P \in V(P_3)$. Obviously, the distance from such vertex to v_P is the same as from v_C to v_P, and at least one n-path connecting this vertex to C contains vertex v_P and all vertices of $V(P_3)$ which are closer to arc C_3. Maximum number of different n-paths containing vertices of $V(P_3)$ at such conditions equals $(\Delta - 1)^{n-1} - 1$. Adding the same numbers from paths P_1 and P_2 together with vertex v_C gives (5) for the maximum possible number of vertices in set $|V_{n3}|$ coinciding with (5). □