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The inverse problem on subset sums[☆]

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ABSTRACT

For a set A, let $P(A)$ be the set of all finite subset sums of A. We prove that if a sequence $B = \{b_1 < b_2 < \cdots\}$ of integers satisfies $b_1 \geq 11$, $b_2 \geq 3b_1+5$, $b_3 \geq 3b_2+3$ and $b_{n+1} > 3b_n - b_{n-2}$ $(n \geq 3)$, then there exists a sequence of positive integers $A = \{a_1 < a_2 < \cdots\}$ such that $P(A) = \mathbb{N} \setminus B$. These lower bounds are optimal in a sense. We pose a problem for further research.

1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of all nonnegative integers. For a sequence of integers $A = \{a_1 < a_2 < \cdots\}$, let

$$P(A) = \{\Sigma \varepsilon_i a_i : a_i \in A, \ \varepsilon_i = 0 \text{ or } 1, \ \Sigma \varepsilon_i < \infty\}.$$

Here we have $0 \in P(A)$. In [1] Burr asked the following question: which sets S of integers are equal to $P(A)$ for some A? He mentioned that if the complement of S grows sufficiently rapidly, then there exists such a sequence A. For example (unpublished), let $B = \{b_1 < b_2 < \cdots\}$ be a sequence of integers for which $b_1 > x_0$ and $b_{n+1} \geq b_n^2$ for all $n \geq 1$, then there exists an A such that $P(A) = \mathbb{N} \setminus B$. In [3] Hegyvári showed that such A exists if $b_1 \geq x_0$ and $b_{n+1} \geq 5b_n$ for all $n \geq 1$. Recently, the first author and Fang [2] showed the following theorems.

Theorem A. *Let $B = \{b_1 < b_2 < \cdots\}$ be a sequence of integers with $b_1 \in \{4, 7, 8\} \cup \{b : b \geq 11, b \in \mathbb{N}\}$ and $b_{n+1} \geq 3b_n + 5$ for all $n \geq 1$. Then there exists a sequence of positive integers $A = \{a_1 < a_2 < \cdots\}$ for which $P(A) = \mathbb{N} \setminus B$.*

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Theorem B. *Let $B = \{b_1 < b_2 < \cdots\}$ be a sequence of integers with $b_1 \in \{3, 5, 6, 9, 10\}$ or $b_2 = 3b_1+4$ or $b_1 = 1$ and $b_2 = 9$ or $b_1 = 2$ and $b_2 = 15$. Then there is no sequence of positive integers $A = \{a_1 < a_2 < \cdots\}$ for which $P(A) = \mathbb{N} \setminus B$.*

Let $[a, b] = \{n : n \in \mathbb{N},\ a \le n \le b\}$ and $x + A = \{x + a : a \in A\}$. In this paper we improve the above results.

Theorem 1. *If $B = \{b_1 < b_2 < \cdots\}$ is a sequence of integers with $b_1 \in \{4, 7, 8\} \cup \{b : b \ge 11,\ b \in \mathbb{N}\}$, $b_2 \ge 3b_1 + 5$, $b_3 \ge 3b_2 + 3$ and $b_{n+1} > 3b_n - b_{n-2}$ for all $n \ge 3$, then there exists a sequence of positive integers $A = \{a_1 < a_2 < \cdots\}$ such that $P(A) = \mathbb{N} \setminus B$ and*

$$P(A_s) = [0, 2b_s] \setminus \{b_1, \ldots, b_s, 2b_s - b_{s-1}, \ldots, 2b_s - b_1\},$$

where $A_s = A \cap [0, b_s - b_{s-1}]$ for all $s \ge 2$.

From the following theorem we see that Theorem 1 is optimal.

Theorem 2. *Let $B = \{b_1 < b_2 < \cdots\}$ be a sequence of integers and $d_1 = 10$, $d_2 = 3b_1+4$, $d_3 = 3b_2+2$ and $d_{n+1} = 3b_n - b_{n-2} (n \ge 3)$. If $b_m = d_m$ for some $m \ge 1$ and $b_n > d_n$ for all $n \ne m$, then there is no sequence of positive integers $A = \{a_1 < a_2 < \cdots\}$ such that*

$$P(A_s) = [0, 2b_s] \setminus \{b_1, \ldots, b_s, 2b_s - b_{s-1}, \ldots, 2b_s - b_1\},$$

where $A_s = A \cap [0, b_s - b_{s-1}]$ for all $s \ge 2$.

Currently we have no answer for the following problem.

Problem 1. Let B and $d_i (i \ge 1)$ be as in Theorem 2 such that $b_m = d_m$ for some $m \ge 3$ and $b_n > d_n$ for all $n \ne m$. Is it true that there is no sequence of positive integers $A = \{a_1 < a_2 < \cdots\}$ with $P(A) = \mathbb{N} \setminus B$?

For $m = 1, 2$, the answer to Problem 1 is affirmative (see [2]).

2. Proof of Theorem 1

By the proof of [2, Theorem 1], there exists a subset A_2 of $[1, b_2 - b_1]$ such that $P(A_2) = [0, 2b_2] \setminus \{b_1, b_2, 2b_2 - b_1\}$. For convenience, let $b_0 = -1$. Then $b_{k+1} > 3b_k - \min\{b_{k-2}, 2b_{k-2}\}$ for all $k \ge 2$.

The set A will be defined by iteration. Assume that $k \ge 2$ and the elements of A_k have been defined for which $A_k = A \cap [0, b_k - b_{k-1}]$ and

$$P(A_k) = [0, 2b_k] \setminus \{b_1, \ldots, b_k, 2b_k - b_{k-1}, \ldots, 2b_k - b_1\}. \quad (1)$$

We deal with the case $k + 1$. If $b_{k+1} \ge 3b_k + 5$, then, by the proof of [2, Theorem 1], we can construct the required A_{k+1} with $A_{k+1} \setminus A_k \subseteq [b_k + 1, b_{k+1} - b_k]$. So we consider the case $3b_k - \min\{b_{k-2}, 2b_{k-2}\} < b_{k+1} < 3b_k + 5$. By $A_k \subseteq [1, b_k - b_{k-1}]$ and (1), we have

$$(b_k - b_{k-2}) + P(A_k) = [b_k - b_{k-2}, 3b_k - b_{k-2}] \setminus B_{k,1},$$

where

$$B_{k,1} = \{b_k - b_{k-2} + b_1, \ldots, \mathbf{b_k} - \mathbf{b_{k-2}} + \mathbf{b_{k-2}}, b_k - b_{k-2} + b_{k-1}, \mathbf{2b_k} - \mathbf{b_{k-2}}, 3b_k - b_{k-2} - b_{k-1}, \ldots, 3b_k - b_{k-2} - b_1\}.$$

Here, for $k = 2$, we have $B_{2,1} = \{b_2 + 1 + b_1, 2b_2 + 1, 3b_2 + 1 - b_1\}$. For $k \ge 3$, we have

$$b_{k-1} < b_k - b_{k-2} + b_1 < \cdots < \mathbf{b_k} - \mathbf{b_{k-2}} + \mathbf{b_{k-2}} < b_k - b_{k-2} + b_{k-1} < 2b_k - b_{k-1} < \mathbf{2b_k} - \mathbf{b_{k-2}} < \cdots < 2b_k - b_1 < 2b_k < 3b_k - b_{k-2} - b_{k-1}.$$

For $k = 2$, we have $b_2 < b_2 + 1 + b_1 < 2b_2 - b_1 < 2b_2 < 3b_2 + 1 - b_1$. Thus, when $k \ge 4$, noting that $b_k - b_{k-2} + b_{k-2}$ is b_k, we have

$$\{b_k - b_{k-2} + b_1, \ldots, b_k - b_{k-2} + b_{k-3}, b_k - b_{k-2} + b_{k-1}\} \subseteq P(A_k)$$