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Inequalities for symmetric means

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ABSTRACT

We study Muirhead-type generalizations of families of inequalities due to Newton, Maclaurin and others. Each family is defined in terms of a commonly used basis of the ring of symmetric functions in n variables. Inequalities corresponding to elementary symmetric functions and power sum symmetric functions are characterized by the same simple poset which generalizes the majorization order. Some analogous results are also obtained for the Schur, homogeneous, and monomial cases.

1. Introduction

Commonly used bases for the vector space Λ_n^r of homogeneous of degree r symmetric functions in n variables $\mathbf{x} = (x_1, \dots, x_n)$ are the *monomial* symmetric functions $\{m_\lambda(\mathbf{x}) \mid \lambda \vdash r\}$, *elementary* symmetric functions $\{e_\lambda(\mathbf{x}) \mid \lambda \vdash r\}$, (complete) *homogeneous* symmetric functions $\{h_\lambda(\mathbf{x}) \mid \lambda \vdash r\}$, *power sum* symmetric functions $\{p_\lambda(\mathbf{x}) \mid \lambda \vdash r\}$, and *Schur* functions $\{s_\lambda(\mathbf{x}) \mid \lambda \vdash r\}$. (See [12, Ch. 7] for definitions.)

To each element $g_\lambda(\mathbf{x})$ of these bases, we will associate a *term-normalized* symmetric function $G_\lambda(\mathbf{x})$ and a *mean* $\mathfrak{G}_\lambda(\mathbf{x})$ by

$$G_\lambda(\mathbf{x}) = \frac{g_\lambda(\mathbf{x})}{g_\lambda(1, \dots, 1)}, \qquad \mathfrak{G}_\lambda(\mathbf{x}) = \sqrt[r]{G_\lambda(\mathbf{x})}. \tag{1.1}$$

Thus, for example $E_\lambda(\mathbf{x})$ and $\mathfrak{E}_\lambda(\mathbf{x})$ are associated with the elementary symmetric function $e_\lambda(\mathbf{x})$. Note that $\{G_\lambda(\mathbf{x}) \mid \lambda \vdash r\}$ forms a basis of Λ_n^r, and that the functions $\{\mathfrak{G}_\lambda(\mathbf{x}) \mid \lambda \vdash r\}$, while symmetric, are not polynomials in $\mathbf{x}$ and therefore do not belong to the *ring of symmetric functions* Λ_n. In the definition of $\mathfrak{G}_\lambda(\mathbf{x})$, we assume $r > 0$.

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The functions $\mathfrak{G}_\lambda(\mathbf{x})$ are examples of *symmetric means* (see, e.g., [2, p. 62]). By definition, these are symmetric functions in $x_1, \dots, x_n$ satisfying

(1) $\min(\mathbf{a}) \le \mathfrak{G}(\mathbf{a}) \le \max(\mathbf{a})$,
(2) $\mathbf{a} \le \mathbf{b}$ (componentwise) implies $\mathfrak{G}(\mathbf{a}) \le \mathfrak{G}(\mathbf{b})$,
(3) $\lim_{\mathbf{b}\to\mathbf{0}} \mathfrak{G}(\mathbf{a}+\mathbf{b}) = \mathfrak{G}(\mathbf{a})$,
(4) $\mathfrak{G}(c\mathbf{a}) = c\mathfrak{G}(\mathbf{a})$,

for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n_{\ge 0}$ and $c \in \mathbb{R}_{\ge 0}$.

This paper will explore inequalities between symmetric means. For fixed n and two means $\mathfrak{F}, \mathfrak{G}$, we will write $\mathfrak{F}(\mathbf{x}) \le \mathfrak{G}(\mathbf{x})$ or $\mathfrak{G}(\mathbf{x}) - \mathfrak{F}(\mathbf{x}) \ge 0$ if we have $\mathfrak{F}(\mathbf{a}) \le \mathfrak{G}(\mathbf{a})$ for all $\mathbf{a} \in \mathbb{R}^n_{\ge 0}$. We define the inequality $F(\mathbf{x}) \le G(\mathbf{x})$ analogously. Note that if the degrees of $F(\mathbf{x})$ and $G(\mathbf{x})$ are equal, then we have $F(\mathbf{x}) \le G(\mathbf{x})$ if and only if $\mathfrak{F}(\mathbf{x}) \le \mathfrak{G}(\mathbf{x})$.

The study of inequalities of symmetric means has a long history. (See, e.g., [2,5].) Perhaps the best known such inequality is that of the *arithmetic* and *geometric* means,

$$\mathfrak{E}_1(\mathbf{x}) \ge \mathfrak{E}_n(\mathbf{x}).$$

See [2] for many proofs of this result. Another example is *Muirhead's inequality* [8]: if λ and μ are partitions of r, then

$$M_\lambda(\mathbf{x}) \le M_\mu(\mathbf{x}) \quad \text{if and only if } \mu \text{ majorizes } \lambda\text{; equivalently,}$$
$$\mathfrak{M}_\lambda(\mathbf{x}) \le \mathfrak{M}_\mu(\mathbf{x}) \quad \text{if and only if } \mu \text{ majorizes } \lambda.$$

See Section 2 for a definition and further discussion of the majorization order (also known as dominance order) on partitions. Muirhead's inequality will serve as a prototype for many of the results in this paper.

Other classical inequalities are due to

(1) Maclaurin [6]: For $1 \le i \le j \le n$,
$$\mathfrak{E}_i(\mathbf{x}) \ge \mathfrak{E}_j(\mathbf{x}),$$
(2) Newton [9, p. 173]: For $1 \le k \le n-1$,
$$E_{k,k}(\mathbf{x}) \ge E_{k+1,k-1}(\mathbf{x}); \quad \text{equivalently,}$$
$$\mathfrak{E}_{k,k}(\mathbf{x}) \ge \mathfrak{E}_{k+1,k-1}(\mathbf{x}),$$
(3) Schlömilch [11]: For $1 \le i \le j$,
$$\mathfrak{P}_i(\mathbf{x}) \le \mathfrak{P}_j(\mathbf{x}),$$
(4) Gantmacher [4, p. 203]: For $k \ge 1$,
$$p_{k,k}(\mathbf{x}) \le p_{k+1,k-1}(\mathbf{x}); \quad \text{equivalently,}$$
$$P_{k,k}(\mathbf{x}) \le P_{k+1,k-1}(\mathbf{x}); \quad \text{equivalently,}$$
$$\mathfrak{P}_{k,k}(\mathbf{x}) \le \mathfrak{P}_{k+1,k-1}(\mathbf{x}),$$
(5) Popoviciu [10]: For $1 \le i \le j$,
$$\mathfrak{H}_i(\mathbf{x}) \le \mathfrak{H}_j(\mathbf{x}),$$
(6) Schur [5, p. 164]: For $k \ge 1$,
$$H_{k,k}(\mathbf{x}) \le H_{k+1,k-1}(\mathbf{x}); \quad \text{equivalently,}$$
$$\mathfrak{H}_{k,k}(\mathbf{x}) \le \mathfrak{H}_{k+1,k-1}(\mathbf{x}).$$

Note that term-normalized symmetric functions and means are defined only for a finite number n of variables. Nevertheless, we may essentially eliminate dependence upon n from the inequalities enumerated above by considering them to be inequalities in sequences of functions,

$$G = (G(x_1), G(x_1, x_2), G(x_1, x_2, x_3), \dots),$$
$$\mathfrak{G} = (\mathfrak{G}(x_1), \mathfrak{G}(x_1, x_2), \mathfrak{G}(x_1, x_2, x_3), \dots).$$