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Small cocircuits in matroids

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This paper is dedicated in memory of Tom Brylawski.

ABSTRACT

We prove that, for any positive integers k, n, and q, if M is a simple matroid that has neither a $U_{2,q+2}$-minor nor an $M(K_n)$-minor and M has sufficiently large rank, then M has a cocircuit of size at most $r(M)/k$.

1. Introduction

The main purpose of this paper is to give simpler proofs of two existing results in extremal matroid theory; we also prove the following new result:

Theorem 1.1. *For any positive integers k, n, and q, there is a positive integer R_1 such that, if M is a simple matroid of rank at least R_1 that has neither a $U_{2,q+2}$-minor nor an $M(K_n)$-minor, then M has a cocircuit of size at most $r(M)/k$.*

This easily implies the main result of [2], as we show immediately below.

Corollary 1.2. *For any positive integers n, k and q, there exists an integer R_2 such that, if M is a simple matroid of rank at least R_2 that has neither a $U_{2,q+2}$-minor nor an $M(K_n)$-minor, then M has a collection of k disjoint cocircuits.*

To prove Corollary 1.2 we use induction on k. The result is trivial for $k = 1$. For $k \geq 2$, we define $R_2(n, k, q) = \max(2R_2(n, k-1, q),\ R_1(n, 2, q))$. Let M be a matroid of rank at least $R_2(n, k, q)$ that has neither a $U_{2,q+2}$-minor nor an $M(K_n)$-minor. We may assume that M is simple. Then, by Theorem 1.1, M has a cocircuit C_k of size at most $r(M)/2$. Thus $r(M/C_k) \geq r(M)/2 \geq R_2(n, k-1, q)$. So, by induction, M/C_k has $k-1$ disjoint cocircuits, say $C_1, \ldots, C_{k-1}$. Thus $C_1, \ldots, C_k$ are disjoint cocircuits in M, as required.

In [3], Corollary 1.2 was used to prove the following result.

Theorem 1.3. *For any positive integers n and q, there exists an integer ρ such that, if M is a simple matroid that has neither a $U_{2,q+2}$-minor nor an $M(K_n)$-minor, then $|E(M)| \leq \rho r(M)$.*

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Note that neither $U_{2,4}$ nor $M(K_5)$ is cographic. Applying Corollary 1.2 to the class of cographic matroids gives the Erdős–Pósa theorem on edge-disjoint circuits in graphs; see [1]. Applying Theorem 1.3 to the class of graphic matroids gives Mader's theorem that, if G is a simple graph with no K_n-minor, then $|E(G)| \leq \rho_n|V(G)|$; see [5].

In this paper we will use the methods of [3] to obtain a new proof of Theorem 1.3 that does not rely on Corollary 1.2. We will then use Theorem 1.3 to prove Theorem 1.1 and, hence, also Corollary 1.2. Proving the results in this order is significantly easier. We use several results from [3,2] but we include their proofs for the sake of completeness.

2. Preliminaries

For a more comprehensive introduction to extremal matroid theory, see the survey paper written by Joseph Kung [4]. We follow the notation of Oxley [6]. A rank-1 flat is referred to as a *point* and a rank-2 flat is referred to as a *line*. The number of points in M is denoted by $\epsilon(M)$. Kung [4] proved the following theorem; we include the proof since it is so nice.

Theorem 2.1. *For any integer $q \geq 2$, if M is a matroid with no $U_{2,q+2}$-minor, then $\epsilon(M) \leq \frac{q^{r(M)}-1}{q-1}$.*

Proof. Let $e \in E(M)$. Inductively we may assume that $\epsilon(M/e) \leq \frac{q^{r(M)-1}-1}{q-1}$. Since e is not in a $(q+2)$-point line, we have

$$\epsilon(M) \leq q\epsilon(M/e) + 1 = q\left(\frac{q^{r(M)-1}-1}{q-1}\right) + 1 = \frac{q^{r(M)}-1}{q-1},$$

as required. $\square$

When q is a prime power, this bound is attained by projective geometries.

Let $\mathcal{U}(q)$ denote the set of all matroids with no $U_{2,q+2}$-minor. Our proof of Theorem 1.3 requires a bound on the number of hyperplanes in a rank-k matroid in $\mathcal{U}(q)$. Fortunately the quality of the bound is not important; we use the following crude upper bound from [3], Proposition 2.3.

Lemma 2.2. *Let $k \geq 1$ and $q \geq 2$ be integers and let $M \in \mathcal{U}(q)$ be a rank-k matroid. Then, M has at most $q^{k(k-1)}$ hyperplanes.*

Proof. Let $n = \epsilon(M)$; thus $n \leq \frac{q^k-1}{q-1} \leq q^k$. Each hyperplane is spanned by $k-1$ points, so the number of hyperplanes is at most $\binom{n}{k-1} \leq n^{k-1} \leq q^{k(k-1)}$. $\square$

The following result is from [2], Lemma 2.3.

Lemma 2.3. *Let $q \geq 2$ be an integer, let $M \in \mathcal{U}(q)$, and let C be a minimum-sized cocircuit of M. If C' is a cocircuit of $M \setminus C$, then $|C'| \geq |C|/q$.*

Proof. Set $F = E(M) - (C \cup C')$. Then F is a flat of M and M/F is a line with at most $q+1$ points. So there are at most $q+1$ hyperplanes of M containing F, one of which is $E(M) - C$. Let the others be $H_1, H_2, \ldots, H_{q'}$. Then $q' \leq q$ and $\{H_1 - F, H_2 - F, \ldots, H_{q'} - F\}$ is a partition of C. Since C is a cocircuit of minimum size,

$$\begin{aligned} q'|C| &\leq \sum_{i=1}^{q'} |E(M) - H_i| \\ &= \sum_{i=1}^{q'} (|C| + |C'| - |H_i - F|) \\ &= q'|C| + q'|C'| - |C|. \end{aligned}$$

Therefore $|C'| \geq |C|/q' \geq |C|/q$. $\square$