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The enumeration of vertex induced subgraphs with respect to the number of components

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ABSTRACT

Inspired by the study of community structure in connection networks, we introduce the graph polynomial $Q(G; x, y)$, the bivariate generating function which counts the number of connected components in induced subgraphs.

We give a recursive definition of $Q(G; x, y)$ using vertex deletion, vertex contraction and deletion of a vertex together with its neighborhood and prove a universality property. We relate $Q(G; x, y)$ to other known graph invariants and graph polynomials, among them partition functions, the Tutte polynomial, the independence and matching polynomials, and the universal edge elimination polynomial introduced by I. Averbouch et al. (2008) [5].

We show that $Q(G; x, y)$ is vertex reconstructible in the sense of Kelly and Ulam, and discuss its use in computing residual connectedness reliability. Finally we show that the computation of $Q(G; x, y)$ is $\sharp\mathbf{P}$-hard, but fixed parameter tractable for graphs of bounded tree-width and clique-width.

1. Introduction

1.1. Motivation: community structure in networks

In the last decade stochastic social networks have been analyzed mathematically from various points of view. Understanding such networks sheds light on many questions arising in biology, epidemiology, sociology and large computer networks. Researchers have concentrated particularly

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on a few properties that seem to be common to many networks: the small-world property, power-law degree distributions, and network transitivity, For a broad view on the structure and dynamics of networks, see [39]. Girvan and Newman [26] highlight another property that is found in many networks: the property of *community structure*, in which network nodes are joined together in tightly knit groups, between which there are only looser connections.

Motivated by [38], and the first author's involvement in a project studying social networks, we were led to study the graph parameter $q_{ij}(G)$, the number of vertex subsets $X \subseteq V$ with i vertices such that $G[X]$ has exactly j components. $q_{ij}(G)$, counts the number of degenerated communities which consist of i members, and which split into j isolated subcommunities.

The ordinary bivariate generating function associated with $q_{ij}(G)$ is the two-variable graph polynomial

$$Q(G; x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n} q_{ij}(G) x^i y^j.$$

We call $Q(G; x, y)$ the *subgraph component polynomial*[1] of G. The coefficient of y^k in $Q(G; x, y)$ is the ordinary generating function for the number of vertex sets that induce a subgraph of G with exactly k components.

The polynomial $Q(G; x, y)$ is also related to measuring graph connectivity in reliability analysis of networks with a given probability of node failures [10,11]. In particular, in [45] the *residual node connectedness reliability of a graph G* is introduced, which can be computed using $Q(G; x, y)$; cf. Section 8.

1.2. $Q(G; x, y)$ as a graph polynomial

There is an abundance of graph polynomials studied in the literature, and a framework is slowly emerging [33,34,27], which allows us to compare graph polynomials with respect to their ability to distinguish graphs, and to encode other graph polynomials or numeric graph invariants, and their computational complexity. In this paper we study the *subgraph component polynomial* $Q(G; x, y)$ as a graph polynomial in its own right and explore its properties within this emerging framework.

Like the bivariate Tutte polynomial (see [12, Chapter 10]), the polynomial $Q(G; x, y)$ has several remarkable properties. However, its distinguishing power is quite different from those of the Tutte polynomial and other well-studied polynomials.

Our main findings are:

- $Q(G; x, y)$ distinguishes graphs which cannot be distinguished by the matching polynomial, the Tutte polynomial, the characteristic polynomial, or the bivariate chromatic polynomial introduced in [20] (Section 3).
- Nevertheless, we construct an infinite family of pairs of graphs which cannot be pairwise distinguished by $Q(G; x, y)$ (Proposition 21).
- The Tutte polynomial satisfies a linear recurrence relation with respect to edge deletion and edge contraction, and is universal in this respect. $Q(G; x, y)$ also satisfies a linear recurrence relation, but with respect to three kinds of vertex elimination, and is universal in this respect (Theorems 13 and 22).
- A graph polynomial in three indeterminates, $\xi(G; x, y, z)$, which satisfies a linear recurrence relation with respect to three types of edge elimination, and which is universal in this respect, was introduced in [4–6]. It subsumes both the Tutte polynomial and the matching polynomial. For a line graph $L(G)$ of a graph G, we have that $Q(L(G); x, y)$ is a substitution instance of $\xi(G; x, y, z)$ (Theorem 23).

[1] Another suitable name would be the *residual component polynomial* to stress the similarity with the polynomial defined in [10].