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Elementary proof techniques for the maximum number of islands

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ABSTRACT

Islands are combinatorial objects that can be intuitively defined on a board consisting of a finite number of cells. It is a fundamental property that two islands are either containing or disjoint. Czédli determined the maximum number of rectangular islands. Pluhár solved the same problem for bricks, and Horváth, Németh and Pluhár for triangular islands. Here, we give a much shorter proof for these results, and also for new, analogous results on toroidal and some other boards.

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1. Introduction and preliminaries

We start with an intuitive notion. Let a rectangular $m \times n$ board be given. We associate a number (real or integer) with each cell of the board. We can think of this number as a height above sea level. A rectangular part of the board is called a *rectangular island* if and only if there is a possible water level such that the rectangle is an island in the usual sense (Fig. 1).

The notion of an island turned up recently in information theory. The characterization of the lexicographical length sequences of binary maximal instantaneous codes in Földes and Singhi [4] uses the notion of *full segments*, which are one-dimensional islands. Several generalizations led to interesting combinatorial problems. Czédli discovered a connection between islands and weakly independent subsets of finite distributive lattices. He determined the maximum number of rectangular islands on a rectangular board [2]. His method is based on weak bases of a finite distributive lattice [3]. Pluhár [7] gave upper and lower bounds in higher dimensions. The third author together with Németh and Pluhár [5] gave upper and lower bounds for the maximum number of triangular islands on a triangular board. Lengvárszky [6] determined the minimal size of a maximal system of islands. In the present paper, we list some related problems with exact formulae. In each case, we present the proof which we believe to be the shortest.

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1	2	1	2
1	5	3	1
2	3	5	1
1	5	3	2
2	1	2	1

Fig. 1. Rectangular landscape with heights.

In full generality, we denote the set of all cells of some board by $\mathcal{C}$. A *height function* is a mapping $h : \mathcal{C} \to \mathbb{R}, c \mapsto h(c)$. We have to specify a neighborhood relation on the cells. If not otherwise stated, two cells are *neighbors* if they share a point. Let R be a subset of cells. The neighbors of R can be defined naturally as the set of cells not in R but having a neighbor in R. A connected subset R of cells is called an *island* if the minimum height in R is greater than the maximum height on the neighbors of R. That is, a water level makes it an island in the usual sense. We always fix a geometric shape, and consider the islands of this shape only. If h is a height function, then we denote the induced set of islands by $\mathcal{I}(h)$. Let us consider rectangular islands. Rectangles R and S are *far from each other* if no cell of R is the neighbor of any cell of S. We denote by $P(\mathcal{C})$ the power set of $\mathcal{C}$, that is the set of all subsets of $\mathcal{C}$. The following statement in a different form was proved in [2].

Lemma 1. *Let $\mathcal{C}$ be the set of all cells of some board, and let $\mathcal{B}$ denote the entire board as an island. Let $\mathcal{I}$ be a set of islands. The following two conditions are equivalent:*

(i) *there exists a mapping $h : \mathcal{C} \to \mathbb{R}, c \mapsto h(c)$ such that $\mathcal{I} = \mathcal{I}(h)$.*
(ii) *$\mathcal{B} \in \mathcal{I}$, and for any $R_1 \neq R_2 \in \mathcal{I}$ either $R_1 \subset R_2$, or $R_2 \subset R_1$, or R_1 and R_2 are far from each other.*

A subset of $P(\mathcal{C})$ satisfying the equivalent conditions of Lemma 1 is called *a system of islands*. The set of maximal elements of $\mathcal{I} \setminus \{\mathcal{B}\}$ is denoted by $\max \mathcal{I}$.

2. Methods

We list three effective techniques of proof for island problems. We give detailed demonstrations of the latter two; the description of the original method can be read in [2]. We recall the following result of [2]:

Lemma 2. *The maximum number of rectangular islands of an $m \times n$ rectangular board is $f(m, n) = [(m+1)(n+1)/2] - 1$.*

Let $\mathcal{C}$ be the set of unit squares of the $m \times n$ board. The proof in [2] exploits that the islands form a weakly independent set in the distributive lattice of $P(\mathcal{C})$. In a distributive lattice, maximal weakly independent subsets are called *weak bases*. By the main theorem of [3], any two weak bases have the same cardinality. The details are given in [2].

We use basic graph theory; see [1]. A graph without a cycle is called a *forest*. A connected forest is a *tree*. A forest with a distinguished node (root) in each component is called a *rooted forest*. If a vertex u is not a root, then u^+ denotes the node following u on the unique path from u to the root. It is called the *father of* u, while u is a *son* of u^+. If v is on the path from u to a root, then v is an *ancestor* of u and u is a *descendant* of v. For any v the vertex v and its descendants span a *rooted subtree* T_v. Finally, *leaves* are vertices without sons. A rooted tree is *binary* if and only if any non-leaf node has precisely two sons.

Let a height function h be defined on the set of cells, fix an island shape, and consider $\mathcal{I}(h)$. In our discussion, the entire board $\mathcal{B}$ is always an island itself. Therefore, the Hasse diagram of $(\mathcal{I}(h), \subseteq)$ is a rooted tree with root $\mathcal{B}$, denoted as $T_0(\mathcal{I}(h))$. The sons of $\mathcal{B}$ are the *maximal* islands, and they are disjoint. The leaves of $T_0(\mathcal{I}(h))$ are the *minimal* islands.

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