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Matching and edge-connectivity in regular graphs

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ABSTRACT

Henning and Yeo proved a lower bound for the minimum size of a maximum matching in a connected k-regular graph with n vertices; it is sharp infinitely often. In an earlier paper, we characterized when equality holds. In this paper, we prove a lower bound for the minimum size of a maximum matching in an l-edge-connected k-regular graph with n vertices, for $l \geq 2$ and $k \geq 4$. Again it is sharp for infinitely many n, and we characterize when equality holds in the bound.

1. Introduction

Petersen [10] proved that every cubic graph with no cut-edges has a perfect matching. It is natural to ask what happens when there are cut-edges. The *matching number* of a graph G, written $\alpha'(G)$, is the maximum size of a matching in G. Biedl et al. [2] determined the smallest matching number among connected cubic graphs with n vertices. Henning and Yeo [6] extended this to connected k-regular n-vertex graphs for appropriate n. In [9], we gave a short proof of their bound for odd k, characterized the extremal graphs, and studied the relationship between the matching number and the number of cut-edges.

Chartrand et al. [4] determined the minimum number of vertices in a k-regular $(k-2)$-edge-connected graph with no perfect matching. Niessen and Randerath [8] extended this to k-regular l-edge-connected graphs. In another direction, Broere et al. [3] gave a formula for the minimum size of a matching among k-regular $(k-2)$-edge-connected graphs with a fixed number of vertices (see also [7]). Our general lower bound for the minimum size of a matching in a k-regular l-edge-connected graph with n vertices implies the results we have mentioned when the parameters are set to appropriate values. Although this bound is sharp infinitely often when $l > 1$, for $l = 1$ the bound in [6,9] is stronger. In Section 3, we characterize the graphs achieving equality in the bound for $l > 1$, and in Section 4 we show that there are infinitely many of them.

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We note also that a forthcoming paper by Cioabă and O [5] explores the relationship among matching, edge-connectivity, and eigenvalues.

2. The lower bound

We use the Berge–Tutte Formula for the matching number. The deficiency def(S) of a vertex set S in G is defined by $\mathrm{def}(S) = o(G-S) - |S|$, where $o(H)$ is the number of odd components in a graph H. Tutte [12] proved that a graph G has a 1-factor if and only if $\mathrm{def}(S) \le 0$ for all $S \in V(G)$. The equivalent Berge–Tutte Formula (see [1]) states that $\alpha'(G) = \min_{S\subseteq V(G)} \frac{1}{2}(n - \mathrm{def}(S))$.

In our counting arguments based on the Berge–Tutte Formula, we consider edge cuts that separate an odd number of vertices from the rest of the graph. Since the degree sum of any graph is even, it follows that for such a cut in a k-regular graph, the size of the cut has the same parity as k. Thus the bound when the edge-connectivity has opposite parity from the degree is the same as the bound for the next larger value of edge-connectivity. That is, it suffices to study $(2t+1)$-edge-connected $(2r+1)$-regular graphs and $2t$-edge-connected $2r$-regular graphs.

Since $2r^2 + r = 2\left(r+\frac{1}{4}\right)^2 - \frac{1}{8}$, the formula in Theorem 2.2 has a very similar flavor to that in Theorem 2.1. In the special case $t = r-1$, the formulas in Theorems 2.1 and 2.2 reduce to essentially the formula in Broere et al. [3]. Also when n is even and less than $2(k\lceil k/2\rceil + k - 1)$, those formulas imply that a $(k-2)$-edge-connected k-regular graph with n vertices has a perfect matching; this is the result of Chartrand et al. [4]. More generally, for l-edge-connected graphs, the threshold on the number of vertices for graphs without perfect matchings given in Niessen and Randerath [8] also follows.

Theorem 2.1. *If G is a $(2t+1)$-edge-connected $(2r+1)$-regular graph with n vertices, where $0 \le t \le r$, then $\alpha'(G) \ge \frac{n}{2} - \left(\frac{r-t}{2(r+1)^2+t}\right)\frac{n}{2}$.*

Proof. Let S be a set with maximum deficiency. Thus, $\alpha'(G) = \frac{1}{2}(n - \mathrm{def}(S))$, where $\mathrm{def}(S) = o(G-S) - |S|$. Let c_i count the odd components of $G-S$ joined to S by exactly i edges; note that c_i is nonzero only when i is odd. Let $c = c_{(2t+1)} + \cdots + c_{(2r-1)}$, and let $c' = o(G-S) - c$. Each odd component counted by c' is joined to S by at least $2r+1$ edges. Note that for $2t+1 \le i \le 2r-1$, each odd component of $G-S$ joined to S by exactly i edges has at least $2r+3$ vertices. (At least $q(2r+2-q)$ edges join any set of q vertices to the rest of G, and this lower bound is at least $2r+1$ when $q \le 2r+1$.)

Since the edges incident to S include the edges joining S to odd components of $G-S$, we have $(2r+1)|S| \ge (2r+1)c' + (2t+1)c$, and hence $|S| \ge c' + \left(\frac{2t+1}{2r+1}\right)c \ge \left(\frac{2t+1}{2r+1}\right)c$. Therefore, $n \ge |S| + c(2r+3) \ge \left(\frac{2t+1}{2r+1} + 2r + 3\right)c$, which yields $c \le \left(\frac{2r+1}{4r^2+8r+4+2t}\right)n$. We compute

$$\begin{aligned}\mathrm{def}(S) &= (c+c') - |S| \le c - \frac{2t+1}{2r+1}c = \frac{2(r-t)}{2r+1}c \\ &\le \frac{2(r-t)}{2r+1}\left(\frac{2r+1}{4r^2+4r+4+2t}\right)n = \frac{(r-t)n}{2(r+1)^2+t}. \quad \square\end{aligned}$$

As noted earlier, the same bound holds for $2t$-edge-connected $(2r+1)$-regular graphs. Similarly, the bound in the next theorem also holds for $(2t-1)$-edge-connected $2r$-regular graphs.

Theorem 2.2. *If G is a $2t$-edge-connected $2r$-regular graph with n vertices, where $1 \le t \le r$ and $r \ge 2$, then $\alpha'(G) \ge \frac{n}{2} - \left(\frac{r-t}{2r^2+r+t}\right)\frac{n}{2}$.*

Proof. The proof is similar to that of Theorem 2.1. Defining S and c_i as in that proof, here the contributions are nonzero only when i is even and at least $2t$. Also, for $2t \le i \le 2r-2$, the odd components of $G-S$ joined to S by i edges have at least $2r+1$ vertices. The same steps as before then lead to $\mathrm{def}(S) \le \frac{(r-t)n}{2r^2+r+t}$. $\square$