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Edge connectivity in difference graphs and some new constructions of partial sum families

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ABSTRACT

In this paper, bounds for the edge connectivity of m-Cayley graphs are found, and also several structural conditions are given for a connected k-regular bi-abelian graph to have edge connectivity strictly less than k. Finally, two infinite families of partial sum families that generate directed strongly regular graphs with new parameters are shown.

1. Introduction

The notion of directed strongly regular graph was introduced by Duval in [5]. A directed graph X without loops, of valency k and order v is called a *directed strongly regular graph with parameters* (v, k, μ, λ, t) (for short, (v, k, μ, λ, t)-DSRG or simply DSRG if we do not specify the parameters) whenever for any vertex u of X there are t undirected edges having u as an endvertex and for every two different vertices u and w of X the number of paths $p(u, w)$ of length 2 starting at u and ending at w depends only on whether uw is an arc of X or not. In particular,

$$p(u, w) = \begin{cases} t & \text{if } u = w \\ \lambda & \text{if } u \neq w \text{ and } uw \in A(X) \\ \mu & \text{if } u \neq w \text{ and } uw \notin A(X) \end{cases}$$

(where $A(X)$ denotes the arc set of X). A directed strongly regular graph will also be refereed to as a *strongly regular digraph*. Some infinite families of directed strongly regular graphs can be found in [6,8,9,11,12].

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Given integers $m \geq 1$ and $n \geq 2$, an automorphism group of a digraph is called (m, n)-*semiregular* if it has m orbits of length n and no other orbit, and the action is regular on each orbit. An m-Cayley digraph $\mathcal{G}$ is a digraph admitting an (m, n)-semiregular group H of automorphisms. When H is abelian, we say that $\mathcal{G}$ is *m-abelian*. If H is generated by an automorphism ρ (that is to say, when H is a cyclic group) and $m = 1$ (respectively, $m = 2$) we say that $\mathcal{G}$ is *n-circulant* (respectively, *n-bicirculant*). Every m-Cayley digraph $\mathcal{G}$ can be represented, following the terminology established by the fourth author in [16], by an $m \times m$ array of subsets of H in the following way. Let $U_0, \ldots, U_{m-1}$ be the m orbits of H, and for each i let $u_i \in U_i$. For each i and j, let $S_{i,j}$ be defined by $S_{i,j} = \{\rho \in H \mid u_i \to \rho(u_j)\}$. The family $(S_{i,j})$ is called the *symbol* of $\mathcal{G}$ relative to $(H; u_0, \ldots, u_{m-1})$.

Now the question of when an m-Cayley digraph is a DSRG arises. Let us remark that the strong regularity question for Cayley and bi-Cayley graphs is related to the notions of partial difference sets [3,14] and partial difference triples [15,4]. Strongly regular tri-Cayley graphs were studied by the second and fourth authors in [13], whereas the strong regularity question for m-Cayley digraphs is related to the notion of partial sum families which was first introduced in [17] by the first and third authors in the context of the so-called difference digraphs, whose definition we recall in the next paragraph. In many cases, we will be dealing with abelian groups, but the definition of difference digraph is generally valid for arbitrary finite groups. In any case, we will use additive notation for the group operation, so that the identity is denoted by 0.

If G is a finite group, H a normal subgroup of G and $\varphi : G/H \times G/H \longrightarrow \mathcal{P}(G - \{0\})$ a mapping satisfying

$$\varphi(\bar{x}, \bar{y}) \subseteq \overline{y - x} \quad \text{for every } \bar{x}, \bar{y} \in G/H,$$

then the *difference digraph* induced by φ is the digraph $\mathcal{G}_{G,H,\varphi} = (G, E)$ with vertex set G and adjacencies satisfying the following condition: $xy \in E$ if $y - x \in \varphi(\bar{x}, \bar{y})$.

An equivalent definition of difference digraphs was also introduced in [17]:

If G is a finite group, H a normal subgroup of G and $\Phi : G/H \longrightarrow \mathcal{P}(G - \{0\})$ is a mapping, then the difference digraph induced by Φ is the digraph $\Gamma_{G,H,\Phi} = (G, E)$ such that $xy \in E$ if $y - x \in \Phi(\bar{x})$.

Given a digraph $\mathcal{G}_{G,H,\varphi}$, we can express it in the form $\Gamma_{G,H,\Phi}$ by taking $\Phi(\bar{x}) = \bigcup_{\bar{y} \in G/H} \varphi(\bar{x}, \bar{y})$ for each coset $\bar{x}$. Similarly, given a digraph $\Gamma_{G,H,\Phi}$, we can express it in the form $\mathcal{G}_{G,H,\varphi}$ by taking $\varphi(\bar{x}, \bar{y}) = \Phi(\bar{x}) \bigcap \overline{y - x} \, \forall \bar{x}, \bar{y} \in G/H$. In some cases the notation $\mathcal{G}_{G,H,\varphi}$ will come handy, as in the definition of partial sum family (Definition 1.1), whereas in other cases we prefer to use the notation $\Gamma_{G,H,\Phi}$, as in most of Section 2.

A very important particular case of the φ construction is when $G = H \times C_m$, where C_m is the cyclic group of order m. In this case, the pairs $(0, i)$, $i \in C_m$ form a transversal for the cosets in G/H, and $\varphi(\overline{(0, i)}, \overline{(0, j)})$ is of the form $S_{i,j} \times (j - i)$, with $S_{i,j} \subseteq H$. Hence it suffices to give the sets $S_{i,j}$ to determine the digraph. It can be easily seen that in this case the difference digraph is an m-Cayley graph of the group H (where m is the index of H in G). Moreover, the sets $S_{i,j}$ form the *symbol* of the digraph as defined above. In the constructions of Section 3 we will consider this case in which $G = H \times C_m$, and hence in that section we will use the symbol notation. As we said before, the problem of characterizing which m-Cayley digraphs are DSRGs was solved in [17] by using the concept of partial sum family. Next, we will give the definition in the terminology of symbols. In the definition, unlike the notation used in [17], we have included m and n in the list of parameters. Condition (iii) of the definition represents an identity in the group ring $\mathbb{Z}[H]$ where, as usual, a subset of H is identified with the sum in $\mathbb{Z}[H]$ of its elements (for definitions and results on group rings, we refer the reader to [18]). In the definition, $\gamma = t - \mu$ and $\beta = \lambda - \mu$.

Definition 1.1. Let H be a group of order n and m an integer with $m \geq 1$. A family $\mathfrak{S} = \{S_{i,j}\}$, with $0 \leq i, j < m$, of subsets of H is an $(m, n, k, \mu, \lambda, t)$-partial sum family (for short, $(m, n, k, \mu, \lambda, t)$-PSF, or simply PSF if we do not specify the parameters) if it satisfies:

(i) $0 \notin S_{i,i}$ for every i.
(ii) $\sum_{j=0}^{m-1} |S_{i,j}| = \sum_{j=0}^{m-1} |S_{j,i}| = k$ for every i.
(iii) $\sum_{k=0}^{m-1} S_{i,k}S_{k,j} = \delta_{i,j}\gamma\{0\} + \beta S_{i,j} + \mu H$ for every i and j, where $\delta_{i,j}$ is the Kronecker delta and 0 is the identity of the group H.