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Combinatorial characterization of the Assur graphs from engineering

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ABSTRACT

We introduce the idea of Assur graphs, a concept originally developed and exclusively employed in the literature of the kinematics community. This paper translates the terminology, questions, methods and conjectures from the kinematics terminology for one degree of freedom linkages to the terminology of Assur graphs as graphs with special properties in rigidity theory. Exploiting the recent works in combinatorial rigidity theory we provide mathematical characterizations of these graphs derived from 'minimal' linkages. With these characterizations, we confirm a series of conjectures posed by Offer Shai, and offer techniques and algorithms to be exploited further in future work.

1. Introduction

Working in the theory of mechanical linkages, the concept of 'Assur groups' was developed by Leonid Assur (1878–1920), a professor at the Saint-Petersburg Polytechnical Institute. In 1914 he published a treatise (reprinted in [2]) entitled *Investigation of plane bar mechanisms with lower pairs from the viewpoint of their structure and classification*. In the kinematics literature it is common to introduce 'Assur groups' (selected groups of links) as special minimal structures of links and joints with zero mobility, from which it is not possible to obtain a simpler substructure of the same mobility [13]. Initially Assur's paper did not receive much attention, but in 1930 the well known kinematician I.I. Artobolevskiĭ, a member of the Russian academy of sciences, adopted Assur's approach and employed it in his widely used book [1]. From that time on Assur groups have been widely employed in Russia and other eastern European countries, while their use in the west is not

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so common. However, from time to time Assur groups are reported in research papers for diverse applications such as: position analysis of mechanisms [11]; finding dead-center positions of planar linkages [13] and others.

The mechanical engineering terminology for linkages (kinematics) and their standard counting techniques are introduced via an example in the next section. Central to Assur's method is the decomposition of complex linkages into fundamental, minimal pieces whose analyses could then be merged to give an overall analysis. Many of these approaches for Assur groups were developed from a range of examples, analyzed geometrically and combinatorially, but never defined with mathematical rigor.

In parallel, rigidity of bar and joint structures as well as motions of related mechanisms have been studied for several centuries by structural engineers and mathematicians. Recently (since 1970) a focused development of a mathematical theory using combinatorial tools was successful in many applications. For example for planar graphs there is a simple geometric duality theory, which, if applied to mechanisms and frameworks yields a relation between statics and kinematics: any locked planar mechanism is dual to an unstable planar isostatic framework (determinate truss) [14,6].

The purpose of this paper is twofold. First, we want to draw together the vocabulary and questions of mechanical engineering with the rigidity theory terminologies of engineering and mathematics. Second, we want to apply the mathematical tools of rigidity theory, including the connections between statics and kinematics, to give precision and new insights into the decomposition and analysis of mechanical linkages.

The mathematical tools we need are briefly sketched with references provided in Sections 2.3–2.7. Our main result is the description of Assur graphs (our term for Assur groups) in Engineering terms (Sections 2.1 and 2.2) and its reformulation in mathematical terms. We show that our mathematical reformulation allows us in a natural way to embed Assur's techniques in the theory of frameworks (Section 3) and bring the results back to linkages. In the process we verify several conjectured characterizations presented by Offer Shai in his talk concerning the generation of Assur graphs and the decomposition of linkages into Assur graphs, at the 2006 Vienna Workshop on Rigidity and Flexibility Section 3.1. We also give algorithmic processes for decomposing general linkages into Assur graphs, as well as for generating all Assur graphs (Sections 3.2 and 3.3).

In a second paper [15], we will apply the geometric theory of bar-and-joint framework rigidity in the plane to explore additional properties and characterizations of Assur graphs. This exploration includes singular (stressed) positions of the frameworks, explored using reciprocal diagrams, and the introduction of 'drivers', which appear in passing in the initial example in the next section.

2. Preliminaries

In the first two sub-sections we present the mechanical engineering vocabulary, problems and approaches through an example. These offer the background and the motivation for the concepts of the paper, but do not yet give the formal mathematical definitions. In the remaining five sub-sections we give the framework basics needed to mathematically describe these approaches.

2.1. Linkages and Assur graphs

A linkage is a mechanism consisting of rigid bodies, the *links*, held together by joints. Since we only consider linkages in the plane, all our joints are pin joints, or pins. A complex linkage may be efficiently studied by decomposing it into simple pieces, the Assur groups. (Engineers use the term group to mean a specified set of links. In mathematics the word group is used for an algebraic structure, but most of the tools we will use come from graph theory, so the word graph seems more natural and we will use it as soon as we start our mathematical section). We introduce these ideas via an example.

Fig. 1 depicts an excavator attached with a linkage system. In the following, we illustrate how the schematic drawing of this system is constructed and how it is decomposed into Assur groups.

In order to get a uniform scheme, termed *structural scheme*, it is common to represent all the connections between the links by revolute joints as appears in Fig. 2. Here joints 0_1 and 0_3 attach the excavator to the vehicle (fixed ground) and these special joints are marked with a small hatched