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Maximal independent sets in bipartite graphs obtained from Boolean lattices

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ABSTRACT

Attempts to enumerate maximal antichains in Boolean lattices give rise to problems involving maximal independent sets in bipartite graphs whose vertex sets are comprised of adjacent levels of the lattice and whose edges correspond to proper containment. In this paper, we find bounds on the numbers of maximal independent sets in these graphs.

1. Introduction

Here we state the main problems, followed by statements of the main results. The notation and terminology are collected in the third subsection.

1.1. Statement of the problems

Some of the earliest investigations of antichains involve Dedekind's Problem [2], dating from 1897, which asks for the cardinality of the free distributive lattice on n generators, or equivalently, the number of antichains $\mathrm{a}(n)$ in the Boolean lattice $\mathbf{2}^n$ of all subsets of an n-element set. Results on Dedekind's problem include explicit computation for small values of n, an asymptotic solution for $\log_2 \mathrm{a}(n)$ obtained by Kleitman [4] and refined by Kleitman and Markowsky [5], and asymptotics for $\mathrm{a}(n)$ due to Korshunov [6] and Sapozhenko [8]. More recently, Kahn [3] has provided alternative proofs for the Kleitman–Markowsky results using entropy arguments.

An antichain is *maximal* if every proper superset contains a comparable pair of elements. Maximal antichains of the Boolean lattice are very familiar objects; indeed, one of the most well-known

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combinatorial results is Sperner's characterization of those maximal antichains of largest cardinality. But what about counting the maximal antichains in $\mathbf{2}^n$? Let $\mathrm{ma}(n)$ be the number of maximal antichains in $\mathbf{2}^n$ and consider this enumeration problem.

Problem 1. Determine the asymptotic value of $\log_2 \mathrm{ma}(n)$.

Our current results on this problem are given below in Theorem 3. The upper and lower bounds for $\log_2 \mathrm{ma}(n)$ are based on quite straightforward observations and differ by a factor of 2.

One approach to Problem 1 is suggested by the methods used to estimate $\mathrm{a}(n)$. Kleitman established that $\log_2 \mathrm{a}(n)$ is asymptotically equal to $\log_2 \binom{n}{\lfloor n/2 \rfloor}$, that is, the log of the number of antichains of $\mathbf{2}^n$ that are contained in a largest level. Korshunov and Sapozhenko obtained the asymptotics for $\mathrm{a}(n)$ by arguing that almost all antichains are contained in the middle three levels of $\mathbf{2}^n$ (cf. [7], Section 1.2). Following this line, let us consider those maximal antichains of $\mathbf{2}^n$ that are confined to consecutive levels in $\mathbf{2}^n$.

For $0 \leq k < n$, let $\mathcal{B}_{n,k}$ denote the bipartite graph of k- and $k+1$-element subsets of an n-set, with adjacency defined by proper containment, and let $\mathrm{mis}(n, k)$ denote the number of maximal independent sets in $\mathcal{B}_{n,k}$. It is not difficult to see that every maximal independent set in $\mathcal{B}_{n,k}$ is a maximal antichain of $\mathbf{2}^n$ and every maximal antichain of $\mathbf{2}^n$ that is contained in $\mathcal{B}_{n,k}$ is a maximal independent set in this graph. We are led to:

Problem 2. Determine the asymptotic value of $\log_2 \mathrm{mis}(n, k)$.

Most of this paper concerns bounding $\log_2 \mathrm{mis}(n, k)$. Ideas for relating maximal antichains in $\mathbf{2}^n$ and maximal independent sets in $\mathcal{B}_{n,k}$ are outlined in the concluding section.

1.2. Statement of the results

Here are the results on Problem 2, first for $k = o(n)$.

Theorem 1. *Let $k = k(n)$ be a function satisfying $k/n \to 0$ as $n \to \infty$. Then*

$$\log_2 \mathrm{mis}(n, k) = (1 + o(1)) \binom{n}{k}.$$

For k a constant proportion of n we have these bounds.

Theorem 2. *Let α be fixed, $0 < \alpha < 1$, and let $k = \alpha n$. Then*

$$\binom{n-1}{k} \leq \log_2 \mathrm{mis}(n, k) \leq 1.3563(1 + o(1)) \binom{n-1}{k},$$

where $o(1) \to 0$ as $n \to \infty$.

While most of the work in this paper is devoted to proving the upper bound, we believe that the lower bound in Theorem 2, more precisely, $(1 + o(1)) \binom{n-1}{k}$, is of the right order for $\log_2 \mathrm{mis}(n, k)$.

Concerning Problem 1, the upper bound in the theorem below follows directly from Kleitman's result for $\mathrm{a}(n)$, and the lower bound in both the following and the preceding theorems follows from a straightforward observation that we make in the next section.

Theorem 3. *For all n,*

$$\binom{n-1}{\lfloor n/2 \rfloor} \leq \log_2 \mathrm{ma}(n) \leq (1 + o(1)) \binom{n}{\lfloor n/2 \rfloor},$$

where $o(1) \to 0$ as $n \to \infty$.