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Discrete piecewise linear functions

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ABSTRACT

The concept of permutograph is introduced and properties of integral functions on permutographs are investigated. The central result characterizes the class of integral functions that are representable as lattice polynomials. This result is used to establish lattice polynomial representations of piecewise linear functions on $\mathbb{R}^d$ and continuous selectors on linear orders.

1. Introduction

We begin with a simple yet instructive example. Let f be a piecewise linear function (PL-function) defined by

$$f(x) = \begin{cases} g_1(x), & \text{for } x \leq -1, \\ g_2(x), & \text{for } -1 \leq x \leq 1, \\ g_3(x), & \text{for } x \geq 1 \end{cases}$$

where

$$g_1(x) = x + 2, \qquad g_2(x) = -x, \qquad g_3(x) = 0.5x - 1.5.$$

The graph of this function is shown in Fig. 1.

It is easy to verify that

$$f = g_1 \wedge (g_2 \vee g_3) = (g_1 \wedge g_2) \vee (g_1 \wedge g_3).$$

We use the notations

$$a \wedge b = \min\{a, b\} \quad \text{and} \quad a \vee b = \max\{a, b\}$$

throughout the article. Thus, the function f can be represented as a lattice polynomial in variables g_1, g_2, and g_3. This is true in general: any continuous PL-function h on a convex domain in $\mathbb{R}^d$ is a lattice polynomial whose variables are linear 'components' of h (Theorem 4.2). In various forms, this result was independently established in [1,20,29] (however, see comments in Section 6, item 1).

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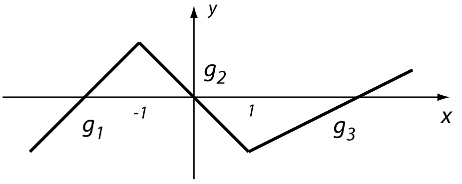

Fig. 1. Graph of a PL-function.

The aim of the article is to show that this result is essentially combinatorial. We introduce a class of functions on permutographs that we call 'discrete piecewise linear (DPL) functions', and show that these functions are representable as lattice polynomials. The discretization of the original problem is achieved by replacing the 'continuity' and 'linearity' properties of PL-functions by the 'separation' and 'linear ordering' properties of integral functions on permutographs.

Permutographs are isometric subgraphs of a weighted Cayley graph of the symmetric group; they are introduced in Section 2. In Section 3, we characterize lattice polynomials on permutographs in terms of the 'separation property' and as 'DPL-functions'. These characterizations are used in Section 4 to establish lattice polynomial representations of PL-functions on convex domains in $\mathbb{R}^d$.

In a different setting, the results of Section 3 are used in Section 5 to obtain a polynomial representation for functions on linear orders. Topological properties of linear orders that are used in Section 5 are introduced in Appendix. Some relevant topics are discussed in Section 6.

2. Permutographs

Let X be a linearly ordered finite set of cardinality $n \geq 1$. We assume that X is the set $\{1, \ldots, n\}$ ordered by the usual relation $<$. A *permutation* (*of order* n) is a bijection $\alpha : X \to X$. We write permutations on the right, that is, $x\alpha$ is the image of x under α, compose them left to right (cf. [7,8]), and use the notation

$$\alpha = (x_1 \cdots x_k \cdots x_n),$$

where $x_k = k\alpha$. For a given permutation $\alpha = (x_1 \cdots x_n)$, the elements of X are linearly ordered by the relation $<_\alpha$ defined by

$$x_i <_\alpha x_j \Longleftrightarrow i < j.$$

In other words,

$$x <_\alpha y \Longleftrightarrow x\alpha^{-1} < y\alpha^{-1}.$$

We write $x \leq_\alpha y$ if $x <_\alpha y$ or $x = y$. Symbols $>_\alpha$ and $\geq_\alpha$ stand for the respective inverse relations.

A pair $\{x, y\}$ of elements of X is called an *inversion* for a pair of permutations $\{\alpha, \beta\}$ if x and y appear in reverse order in α and β. The *distance* $d(\alpha, \beta)$ between permutations α and β is defined as the number of inversions for the pair $\{\alpha, \beta\}$. This distance equals one half of the cardinality of the symmetric difference of the binary relations $<_\alpha$ and $<_\beta$. We say that a permutation γ *lies between* permutations α and β if

$$d(\alpha, \gamma) + d(\gamma, \beta) = d(\alpha, \beta).$$

It is straightforward to see that γ lies between α and β if and only if

$$(x <_\alpha y \text{ and } x <_\beta y) \Rightarrow x <_\gamma y \quad \text{for all } x, y \in X.$$

The set of all permutations of X forms the *symmetric group* S_n with the operation of composition and the identity element $\epsilon = (1 \cdots n)$.

A partition $\pi = (X_1, \ldots, X_m)$ of the ordered set $(X, <)$ into a family of nonempty subsets is said to be an *ordered partition* if

$$(x \in X_i, y \in X_j, i < j) \Rightarrow x < y.$$

The ordered partition $(\{1\}, \ldots, \{n\})$ is said to be *trivial.*