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# On lattices whose minimal vectors form a 6-design

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Dedicated to Eiichi Bannai on the occasion of his 60th birthday

ABSTRACT

Let $L$ be a lattice of dimension $n \leq 24$ such that the minimal vectors of $L$ form a 6-design and generate $L$. Then $L$ is similar to either the root lattice $E_8$, the Barnes–Wall lattice $\mathrm{BW}_{16}$, the Leech lattice $\Lambda_{24}$, or $n = 23$. For $n = 23$ we conjecture that the only possibilities for $L$ are the shorter Leech lattice $O_{23}$ or its even sublattice $\Lambda_{23}$.



## 1. Introduction

Spherical designs were introduced in 1977 by Delsarte, Goethals and Seidel [11] and soon afterwards studied by Eiichi Bannai in a series of papers ([3–5], to mention only a few of them). A spherical $t$-design is a finite subset $X$ of the sphere such that every polynomial on $\mathbb{R}^n$ of total degree at most $t$ has the same average over $X$ as over the entire sphere. The theory of lattices has been used quite successfully to classify good designs of minimal possible cardinality (see [6]). In this paper we use the theory of designs to construct good lattices.

**Definition 1.1.** A t-design-lattice is a lattice $\Lambda$ in Euclidean space such that its minimal vectors

$$\mathrm{Min}(\Lambda) := \{\lambda \in \Lambda \mid (\lambda, \lambda) = \min(\Lambda)\}$$

form a spherical $t$-design and generate the lattice $\Lambda$.

Clearly any $t$-design-lattice is also a $t'$-design-lattice for all $t' \leq t$. Note that the 4-design-lattices are exactly the strongly perfect lattices defined in [14] that are generated by their minimal vectors. They are now classified up to dimension 12 (see [14,12,13]). From this classification we see:

**Theorem 1.2.** *Let $t \geq 4$ be even and let $\Lambda$ be a $t$-design-lattice of dimension $n \leq 12$. Then one of the following holds:*

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(a) *$n = 1$ and $\Lambda$ is similar to $\mathbb{Z}$. Here $t$ is arbitrary since the 0-dimensional sphere $S^0$ consists only of the two minimal vectors $\{1, -1\}$ of $\mathbb{Z}$.*
(b) *$n = 2$, $\Lambda$ is similar to the hexagonal lattice $A_2$, and $t = 4$.*
(c) *$n = 4$, $\Lambda$ is similar to the root lattice $D_4$, and $t = 4$.*
(d) *$n = 6$, $\Lambda$ is similar to the root lattice $E_6$ or its dual lattice $E_6^*$, and $t = 4$.*
(e) *$n = 7$, $\Lambda$ is similar to the root lattice $E_7$ or its dual lattice $E_7^*$, and $t = 4$.*
(f) *$n = 8$, $\Lambda$ is similar to the root lattice $E_8$, and $t \leq 6$.*
(g) *$n = 10$, $\Lambda$ is similar to the lattice $K'_{10}$ or its dual lattice $(K'_{10})^*$, and $t = 4$.*
(h) *$n = 12$, $\Lambda$ is similar to the Coxeter–Todd lattice $K_{12}$, and $t = 4$.*

This paper classifies the 6-design-lattices of dimension $23 \neq n \leq 24$. We will show the following theorem.

**Theorem 1.3.** *Let $t \geq 6$ be even and let $\Lambda$ be a $t$-design-lattice of dimension $n \leq 24$. Then one of the following holds:*

(a) *$n = 1$ and $\Lambda$ is similar to $\mathbb{Z}$.*
(b) *$n = 8$, $\Lambda$ is similar to the root lattice $E_8$, and $t = 6$.*
(c) *$n = 16$, $\Lambda$ is similar to the Barnes–Wall lattice $\mathrm{BW}_{16}$, and $t = 6$.*
(d) *$n = 23$ and $t = 6$. In this dimension there are at least two 6-design-lattices, namely the shorter Leech lattice $O_{23}$ and its even sublattice $\Lambda_{23}$.*
(e) *$n = 24$, $\Lambda$ is similar to the Leech lattice $\Lambda_{24}$, and $t \leq 10$.*

In fact all layers of the lattices in Theorem 1.3 are spherical $t$-designs. This is trivial in case (a) and follows from [2, Corollary 3.1] for the remaining cases except for case (d). For case (d) note that the automorphism group of $O_{23}$ and $\Lambda_{23}$ is $C_2 \times Co_2$ and its first harmonic invariant has degree 8.

We also remark that it is still unknown, whether there are $t$-design-lattices for $t \geq 12$. The only known 10-design-lattices are the known extremal even unimodular lattices of dimension a multiple of 24, namely the Leech lattice $\Lambda_{24}$ and the three unimodular lattices $P_{48p}$, $P_{48q}$ and $P_{48n}$ of dimension 48 with minimum 6 (see [10]).

## 2. Some general remarks on antipodal *t*-designs

In the following we assume that $n \geq 2$ to avoid trivialities. Let $X \subset S^{n-1}$ be a finite subset of the $(n-1)$-dimensional unit-sphere such that $X \cap -X = \emptyset$. For any even number $t = 2h$, the condition that $X \cup -X$ be a spherical $t$-design is equivalent to the existence of some number $c_t$ such that for all $\alpha \in \mathbb{R}^n$

$$(Dt)(\alpha) : \sum_{x \in X} (x, \alpha)^t = c_t |X| (\alpha, \alpha)^h.$$

The constant $c_t$ is then uniquely determined and easily calculated by applying $t$ times the Laplace operator $\Delta$ with respect to $\alpha$ (see [14]) as

$$c_t = \prod_{j=1}^{h} \frac{2j-1}{n+2j-2} \quad \text{(where } t = 2h\text{)}.$$

Note that

$$\Delta(Dt)(\alpha) = (D(t-2))(\alpha).$$

If we apply these equalities to the minimal vectors $X \dot{\cup} -X = \mathrm{Min}(\Lambda)$ of a $t$-design-lattice $\Lambda$ and some minimal vector $\alpha \in \mathrm{Min}(\Lambda^*)$ of the dual lattice we get lower bounds on the Bergé–Martinet invariant

$$\gamma'(\Lambda)^2 := \gamma(\Lambda)\gamma(\Lambda^*) = \min(\Lambda)\min(\Lambda^*)$$

of a $t$-design-lattice as follows.