European Journal of Combinatorics 30 (2009) 1194–1204

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Netlike partial cubes, IV: Fixed finite subgraph theorems

Norbert Polat

I.A.E., Université Jean Moulin (Lyon 3), 6 cours Albert Thomas, 69355 Lyon Cedex 08, France

ARTICLE INFO

Article history:
Available online 4 October 2008

ABSTRACT

We prove that, if a netlike partial cube *G* (see [N. Polat, Netlike partial cubes I. General properties, Discrete Math. 307 (2007) 2704–2722]) contains no isometric rays, then there exists a convex cycle or a finite hypercube which is fixed by every automorphism of *G*. Furthermore we prove that every self-contraction (map which preserves or collapses the edges) of *G* fixes a convex cycle or a finite hypercube if and only if *G* contains no isometric rays. We also study the self-contractions of *G* which fix no finite set of vertices.

1. Introduction

The class of netlike partial cubes was introduced in Part I [16] of this series of papers as a class of partial cubes containing median graphs, even cycles, benzenoid graphs and cellular bipartite graphs as particular elements.

In this fourth paper we pursue the study of netlike partial cubes by focusing on fixed subgraph properties, and chiefly by generalizing three results of Tardif [18] on median graphs. Fixed subgraph theorems are far-reaching outgrowths of metric fixed point theory. They have been a flourishing topic in the recent literature on metric graph theory. See in particular the study [4] by Brešar et al. of tree-like partial cubes, another class of finite partial cubes that contains all finite median graphs.

For a netlike partial cube *G*, just as for median graphs, the property that every self-contraction fixes a finite regular netlike subgraph is directly linked to the absence of isometric rays in *G*. The proofs of this result and of related ones, which form the best part of this paper, require the geodesic topology, a topology which was introduced in [11] for the study of graphs containing no isometric rays, and which turns out for netlike partial cubes to be the topology generated by the convex sets as a subbase.

In the last section we use this topology to specify which ends of a netlike partial cube are directions of translating self-contractions of this graph, namely self-contractions which fix no finite set of vertices.

E-mail address: norbert.polat@univ-lyon3.fr.

0195-6698/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ejc.2008.09.012

2. Preliminaries

2.1. Graphs

The graphs we consider are undirected, without loops or multiple edges, and may be finite or infinite. Let G be a graph. If $x \in V(G)$, the set $N_G(x) := \{y \in V(G) : xy \in E(G)\}$ is the *neighborhood* of x in G, $N_G[x] := \{x\} \cup N_G(x)$ is the *closed neighborhood* of x in G and $\delta_G(x) := |N_G(x)|$ is the *degree* of x in G. For a set X of vertices of G we put $N_G[X] := \bigcup_{x \in X} N_G[x]$ and $N_G(X) := N_G[X] - X$, we denote by $G[X]$ the subgraph of G induced by X, and we set $G - X := G[V(G) - X]$.

A *path* $P = \langle x_0, \ldots, x_n \rangle$ is a graph with $V(P) = \{x_0, \ldots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(P) = \{x_i x_{i+1} : 0 \leq i < n\}$. A path $P = \langle x_0, \ldots, x_n \rangle$ is called an (x_0, x_n)*-path*, x_0 and x_n are its *endvertices*, while the other vertices are called its *inner* vertices, $n = |E(P)|$ is the *length* of P.

A *cycle* C with $V(C) = \{x_1, \ldots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(C) = \{x_i x_{i+1} : 1 \leq i < n\} \cup \{x_n x_1\}$, is denoted by $\langle x_1, \ldots, x_n, x_1 \rangle$. The non-negative integer $n = |E(C)|$ is the *length* of C, and a cycle of length n is called an *n-cycle* and is often denoted by C_n.

Let G be a connected graph. The usual *distance* between two vertices x and y, that is, the length of any (x, y)*-geodesic* (=shortest (x, y)-path) in G, is denoted by $d_G(x, y)$. A connected subgraph H of G is *isometric* in G if $d_H(x, y) = d_G(x, y)$ for all vertices x and y of H. The *(geodesic) interval* $I_G(x, y)$ between two vertices x and y of G is the set of vertices of all (x, y)-geodesics in G.

2.2. Convexities

A *convexity* on a set X is an algebraic closure system $\mathcal{C}$ on X. The elements of $\mathcal{C}$ are the *convex sets* and the pair $(X, \mathcal{C})$ is called a *convex structure*. See van de Vel [19] for a detailed study of abstract convex structures. Several kinds of graph convexities, that is convexities on the vertex set of a graph G, have already been investigated. We will principally work with the *geodesic convexity*, that is the convexity on $V(G)$ which is induced by the geodesic interval operator I_G. In this convexity, a subset C of $V(G)$ is convex provided it contains the geodesic interval $I_G(x, y)$ for all $x, y \in C$. The *convex hull* $co_{\mathcal{G}}(A)$ of a subset A of $V(G)$ is the smallest convex set which contains A. The convex hull of a finite set is called a *polytope*. A subset H of $V(G)$ is a *half-space* if H and $V(G) - H$ are convex. We denote by $\mathcal{I}_G$ the pre-hull operator of the geodesic convex structure of G, i.e. the self-map of $\mathcal{P}(V(G))$ such that $\mathcal{I}_G(A) := \bigcup_{x,y \in A} I_G(x, y)$ for each $A \subseteq V(G)$. The convex hull of a set $A \subseteq V(G)$ is then $co_G(A) = \bigcup_{n \in \mathbb{N}} \mathcal{I}_G^n(A)$. Furthermore we say that a subgraph of a graph G is *convex* if its vertex set is convex, and by the *convex hull* $co_G(H)$ of a subgraph H of G we mean the smallest convex subgraph of G containing H as a subgraph, that is

$$co_G(H) := G[co_G(V(H))].$$

2.3. Netlike partial cubes

First we recall some properties of *partial cubes*, that is of isometric subgraphs of hypercubes. Partial cubes are particular connected bipartite graphs.

For an edge ab of a graph G, let

$$W_{ab}^G := \{x \in V(G) : d_G(a, x) < d_G(b, x)\},$$
$$U_{ab}^G := N_G(W_{ba}^G).$$

If no confusion is likely, we will simply denote W_{ab}^G and U_{ab}^G by W_{ab} and U_{ab}, respectively. Note that the sets W_{ab} and W_{ba} are disjoint and that $V(G) = W_{ab} \cup W_{ab}$ if G is bipartite and connected.

Two edges xy and uv are in the Djoković–Winkler relation Θ if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u).$$

If G is bipartite, the edges xy and uv are in relation Θ if and only if $d_G(x, u) = d_G(y, v)$ and $d_G(x, v) = d_G(y, u)$. The relation Θ is clearly reflexive and symmetric.