European Journal of Combinatorics 30 (2009) 1263–1270

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Hexavalent half-arc-transitive graphs of order $4p$

Xiuyun Wang, Yan-Quan Feng

Department of Mathematics, Beijing Jiaotong University, Beijing 100044, PR China

ARTICLE INFO

Article history:
Received 5 July 2008
Accepted 20 November 2008
Available online 13 January 2009

ABSTRACT

A graph is half-arc-transitive if its automorphism group acts transitively on its vertex set and edge set, but not arc set. It was shown by [Y.-Q. Feng, K.S. Wang, C.X. Zhou, Tetravalent half-transitive graphs of order 4p, European J. Combin. 28 (2007) 726–733] that all tetravalent half-arc-transitive graphs of order $4p$ for a prime p are non-Cayley and such graphs exist if and only if $p-1$ is divisible by 8. In this paper, it is proved that each hexavalent half-arc-transitive graph of order $4p$ is a Cayley graph and such a graph exists if and only if $p-1$ is divisible by 12, which is unique for a given order. This result contributes to the classification of half-arc-transitive graphs of order $4p$ of general valencies.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple and undirected, but with an implicit orientation of the edges when appropriate. For a graph X, let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ be the vertex set, the edge set, the arc set and the automorphism group of X, respectively. Let D_{2n} be the dihedral group of order $2n$, and $\mathbb{Z}_n$ the cyclic group of order n as well as the ring of integers modulo n. Denote by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n, and for a prime p, denote by $\mathbb{Z}_p^m$ the elementary abelian group $\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ (m times). For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. A graph X is isomorphic to a Cayley graph on G if and only if its automorphism group $\mathrm{Aut}(X)$ has a subgroup isomorphic to G, acting regularly on vertices (see [1, Lemma 16.3]).

A graph X is said to be *vertex-transitive*, *edge-transitive* or *arc-transitive* if $\mathrm{Aut}(X)$ acts transitively on $V(X)$, $E(X)$, or $A(X)$, respectively. A graph is said to be *half-arc-transitive* provided that it is vertex-transitive and edge-transitive, but not arc-transitive. More generally, by a half-arc-transitive action of

E-mail address: yqfeng@bjtu.edu.cn (Y.-Q. Feng).

0195-6698/$ – see front matter
doi:10.1016/j.ejc.2008.11.013

a subgroup G of $\mathrm{Aut}(X)$ on a graph X we shall mean a vertex-transitive and edge-transitive, but not arc-transitive action of G on X. In this case, we shall say that the graph X is *G-half-arc-transitive*.

The investigation of half-arc-transitive graphs was initiated by Tutte [2] and he proved that a vertex- and edge-transitive graph with odd valency must be arc-transitive. In 1970 Bouwer [3] constructed a $2k$-valent half-arc-transitive graph for every $k \geq 2$ and later more such graphs were constructed (see [4–10]). Let p be a prime. It is well known that there are no half-arc-transitive graphs of order p or p^2 [11], and by Cheng and Oxley [12], there are no half-arc-transitive graphs of order $2p$. Alspach and Xu [4] classified half-arc-transitive graphs of order $3p$ and Wang [10] classified half-arc-transitive graphs of order a product of two distinct primes. Despite all of these efforts, however, more classifications of half-arc-transitive graphs with general valencies seem to be very difficult. In fact, constructing and characterizing half-arc-transitive graphs with small valencies is currently an active topic in algebraic graph theory (see [13–23]). It was shown in [24] that there are no tetravalent half-arc-transitive Cayley graphs of order $4p$, and a tetravalent half-arc-transitive non-Cayley graph of order $4p$ exists if and only if $p-1$ is divisible by 8, and then such a graph is unique for given order. Opposite to the tetravalent case, in this paper it is proved that each hexavalent half-arc-transitive graph of order $4p$ is a Cayley graph. Such a graph exists if and only if $p-1$ is divisible by 12, and again, such a graph is unique for given order. The result can be helpful to classify half-arc-transitive graphs of order $4p$ (for general valences), a problem that has been considered since 1994 by many authors.

2. Preliminary results

Let $\mathrm{Cay}(G, S)$ be a Cayley graph. Given $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg$, $x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$ is a permutation group isomorphic to G, called the *right regular representation* of G. The Cayley graph $\mathrm{Cay}(G, S)$ is vertex-transitive because it admits $R(G)$ as a regular subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Furthermore, the group $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is also a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Actually, $\mathrm{Aut}(G, S)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))_1$, the stabilizer of the vertex 1 in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $\mathrm{Aut}(\mathrm{Cay}(G, S))$ contains $R(G)$ as a normal subgroup. The following proposition is fundamental for normal Cayley graphs.

Proposition 2.1 (*[22, Proposition 1.5]*). *Let $X = \mathrm{Cay}(G, S)$ be a Cayley graph on a finite group G with respect to S. Let $A = \mathrm{Aut}(X)$ and let A_1 be the stabilizer of 1 in A. Then X is normal if and only if $A_1 = \mathrm{Aut}(G, S)$.*

Now we state a simple observation about half-arc-transitive graphs (see [13]).

Proposition 2.2. *There are no half-arc-transitive graphs with fewer than* 27 *vertices.*

The following proposition is straightforward (see [24, Propositions 2.1 and 2.2]).

Proposition 2.3. *Let $X = \mathrm{Cay}(G, S)$ be a half-arc-transitive graph. Then, there is no involution in S, and no $\alpha \in \mathrm{Aut}(G, S)$ such that $s^\alpha = s^{-1}$ for some $s \in S$. In particular, there are no half-arc-transitive Cayley graphs on abelian groups.*

Li et al. [7] considered primitive half-arc-transitive graphs.

Proposition 2.4 (*[7, Theorem 1.4]*). *There are no vertex-primitive half-arc-transitive graphs of valency less than 10.*

To state the classification of connected cubic and hexavalent symmetric graphs of order $2p$, p a prime, due to Cheng and Oxley [12], we need to define the following graphs. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V = \{i \mid i \in \mathbb{Z}_p\}$ and $V' = \{i' \mid i \in \mathbb{Z}_p\}$. Let r be a positive integer dividing $p-1$ and $H(p, r)$ the unique subgroup of $\mathbb{Z}_p^*$ of order r. Define the graph $G(2p, r)$ to have vertex set $V \cup V'$ and edge set $\{xy' \mid x, y \in \mathbb{Z}_p, y - x \in H(p, r)\}$. Note that $G(2p, p-1) \cong K_{p,p}$. Let τ and ρ be the maps defined as following: $i^\tau = i + 1$ and $i'^\tau = i' + 1$, and $i^\rho = (-i)'$ and $i'^\rho = -i$. Then, the graph $G(2p, r)$ is a Cayley graph since $\langle \tau, \rho \rangle$ is regular on $V(G(2p, r))$.