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The homology of the cycle matroid of a coned graph

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Abstract

The cone $\hat{G}$ of a finite graph G is obtained by adding a new vertex p, called the cone point, and joining each vertex of G to p by a simple edge. We show that the rank of the reduced homology of the independent set complex of the cycle matroid of $\hat{G}$ is the cardinality of the set of the edge-rooted forests in the base graph G. We also show that there is a basis for this homology group such that the action of the automorphism group $\mathrm{Aut}(G)$ on this homology is isomorphic (up to sign) to that on the set of the edge-rooted forests in G.

1. Introduction

Let G be a finite graph and let $I(G)$ denote the independent set complex of the cycle matroid of G: $I(G)$ is a simplicial complex with the vertex set $E(G)$, the set of edges of G, and $\sigma \subset E(G)$ is a face of $I(G)$ if σ does not contain any cycle. An important property of $I(G)$ is shellability, which implies that $I(G)$ has the homotopy type of a wedge of $(n-2)$-dimensional spheres, where n is the order of G. Therefore, a topological invariant that one naturally associates with $I(G)$ is the rank of the top reduced homology group $\tilde{H}_{n-2}(I(G))$. We will denote this invariant by $\alpha(G)$ throughout the paper.

It is well known that when G is connected, $\alpha(G)$ is the number of the spanning trees in G with internal activity zero (refer to [1, Theorem 7.8.1] for a matroid theoretic generalization of this result). This result is somewhat unnatural in that the internal activity of a spanning tree in G depends on the ordering of $E(G)$, whereas $\alpha(G)$ is independent of such orderings. However, for the complete graph K_n, a new combinatorial interpretation for $\alpha(K_n)$ that is independent of any ordering of $E(K_n)$ was found in [3]: $\alpha(K_n)$ is the cardinality of the set of all edge-rooted forests in K_{n-1} (refer to Section 2 for the definition of the edge-rooted forests). Moreover, it

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was shown that one can associate an edge-rooted forest in K_{n-1} with a fundamental cycle in $\tilde{H}_{n-2}(I(K_n))$, the collection of which then forms a basis for $\tilde{H}_{n-2}(I(K_n))$. It is also worth noting that a formula for $\alpha(K_n)$ is known in terms of the Hermite polynomial, a generating function for partial matchings (degree 1 subgraphs) in a graph (refer to [6]).

In this paper, we will generalize the results in [3] to the following class of graphs: define the *cone* $\hat{G}$ of a finite graph G to be the graph obtained by adding a new vertex p, called the *cone point*, and joining each vertex of G to p by a simple edge. For example, the complete graph K_{n+1} is the cone of K_n and the wheel W_n is the cone of the circuit C_n. The first main result of this paper is that $\alpha(\hat{G})$ is again the cardinality of the set $\mathcal{F}_e(G)$ of all edge-rooted forests in G. We will show this by constructing a bijection between $\mathcal{F}_e(G)$ and the set of all spanning trees in $\hat{G}$ with internal activity zero. Also we will create a set of fundamental cycles in $\tilde{H}_{n-1}(I(\hat{G}))$ that is naturally associated with $\mathcal{F}_e(G)$. By means of a partial ordering in $\mathcal{F}_e(G)$ which we will discuss in Section 2.2, we will see that this set forms a basis for $\tilde{H}_{n-1}(I(\hat{G}))$. Most importantly, we will also establish that, when G is simple, the action of the automorphism group $\mathrm{Aut}(G)$ on $\tilde{H}_{n-1}(I(\hat{G}))$ is isomorphic to the permutation action on $\mathcal{F}_e(G)$ tensored with the sign representation.

Throughout the paper, we will assume that G is a non-empty finite graph with the vertex set $[n] = \{1, \ldots, n\}$ when the order of G is $n > 0$. Multiple edges are allowed in G unless otherwise stated. An m-dimensional face $\sigma \in I(G)$ is identified with the spanning forest $F = F(\sigma)$ of G (hence $V(F) = V(G)$) whose edges are the $m + 1$ elements of σ. A face σ of $I(G)$ and the corresponding spanning forest $F = F(\sigma)$ will both be denoted by F. When G is connected, a facet (a maximal simplex under inclusion) of $I(G)$ is a spanning tree in G.

2. Edge-rooted forests

We refer the reader to [7] for matroids in general and the excellent article [1] for the definitions and results concerning independent set complexes of matroids that will be used throughout this paper.

2.1. A new combinatorial interpretation for $\alpha(\hat{G})$

First, we review internal activity. Let G be a finite graph equipped with a linear ordering ω of $E(G)$ the set of its edges. Given a spanning tree T in G, deleting an edge e in T creates a forest with two components, say T_1 and T_2. The *fundamental bond* of e with respect to T is the set $E_G(T_1, T_2)$ of all edges in G having one vertex in T_1 and the other in T_2. In particular, e is always in its own fundamental bond. The edge $e \in T$ is said to be *internally active* if e is ω-smallest in its fundamental bond. Otherwise e is *internally passive*. The internal activity of T is the number of internally active edges in T.

Recall that the *cone* $\hat{G}$ of a graph G is obtained by adding a cone point p to G, and joining each vertex of G to p by a simple edge. For the coned graph $\hat{G}$, we will use the following ordering of $E(\hat{G})$ to determine the spanning trees in $\hat{G}$ with internal activity zero. Let $V(G) = [n]$, and let the vertices of $\hat{G}$ be linearly ordered by $p < 1 < 2 < \cdots < n$. Let ω be the resulting lexicographic ordering of $E(\hat{G})$: $\{p, 1\} <_\omega \{p, 2\} <_\omega \cdots <_\omega \{p, n\} <_\omega \cdots <_\omega \{n - 1, n\}$. Hence every edge in the base G is larger than the edges in the star of p in this ordering.

Now given any spanning tree T in $\hat{G}$, define the *support* of T to be $T \cap G$. Note that $T \cap G$ is a spanning forest in G, and there is exactly one vertex in each component of $T \cap G$ that is adjacent to p in T. This vertex will be called a *connecting root*. The following lemma is a key result of this section, whose proof is immediate and will be omitted.