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The structure of 3-connected matroids of path width three

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Abstract

A 3-connected matroid M is sequential or has path width 3 if its ground set $E(M)$ has a sequential ordering, that is, an ordering $(e_1, e_2, \dots, e_n)$ such that $(\{e_1, e_2, \dots, e_k\}, \{e_{k+1}, e_{k+2}, \dots, e_n\})$ is a 3-separation for all k in $\{3, 4, \dots, n-3\}$. In this paper, we consider the possible sequential orderings that such a matroid can have. In particular, we prove that M essentially has two fixed ends, each of which is a maximal segment, a maximal cosegment, or a maximal fan. We also identify the possible structures in M that account for different sequential orderings of $E(M)$. These results rely on an earlier paper of the authors that describes the structure of equivalent non-sequential 3-separations in a 3-connected matroid. Those results are extended here to describe the structure of equivalent sequential 3-separations.

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1. Introduction

The matroid terminology used here will follow Oxley [3]. Let M be a matroid. When M is 2-connected, Cunningham and Edmonds [1] gave a tree decomposition of M that displays all of its 2-separations. Now suppose that M is 3-connected. Oxley et al. [5] showed that there is a corresponding tree decomposition of M that displays all non-sequential 3-separations of M up to a certain natural equivalence. Both this equivalence and the definition of a non-sequential 3-separation are based on the notion of full closure in M. For a set Y, if Y equals its closure in both

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M and M^*, we say that Y is *fully closed* in M. The *full closure*, $\mathrm{fcl}(Y)$, of Y is the intersection of all fully closed sets containing Y. It is obtained by beginning with Y and alternately applying the closure operators of M and M^* until no new elements can be added. If (X, Y) is a 3-separation of M, then (X, Y) is *sequential* if $\mathrm{fcl}(X)$ or $\mathrm{fcl}(Y)$ is $E(M)$. Two 3-separations (Y_1, Y_2) and (Z_1, Z_2) of M are *equivalent* if $\{\mathrm{fcl}(Y_1), \mathrm{fcl}(Y_2)\} = \{\mathrm{fcl}(Z_1), \mathrm{fcl}(Z_2)\}$.

While the introduction of this notion of equivalence is an essential tool in proving the main result of [5], this equivalence ignores some of the finer structure of the matroid. Hall et al. [2] made a detailed examination of this equivalence and described precisely what substructures in the matroid result in two non-sequential 3-separations being equivalent. The assumption that the 3-separations are non-sequential is helpful in that it gives two fixed ends for the 3-separations in an equivalence class $\mathcal{K}$. More precisely, if (A_1, B_1) is in $\mathcal{K}$, then $(A_1 - \mathrm{fcl}(B_1), \mathrm{fcl}(B_1))$ and $(\mathrm{fcl}(A_1), B_1 - \mathrm{fcl}(A_1))$ are also in $\mathcal{K}$. Letting $A = A_1 - \mathrm{fcl}(B_1)$ and $B = B_1 - \mathrm{fcl}(A_1)$, we have, for every 3-separation (A_2, B_2) in $\mathcal{K}$, that $\{A_2 - \mathrm{fcl}(B_2), B_2 - \mathrm{fcl}(A_2)\} = \{A, B\}$. Thus we can view A and B as the fixed ends of the members of the equivalence class $\mathcal{K}$. Moreover, we can associate with $\mathcal{K}$ a sequence $(A, x_1, x_2, \ldots, x_n, B)$ where $E(M) = A \cup \{x_1, x_2, \ldots, x_n\} \cup B$ and, for all i in $\{0, 1, \ldots, n\}$, the partition $(A \cup \{x_1, x_2, \ldots, x_i\}, \{x_{i+1}, x_{i+2}, \ldots, x_n\} \cup B)$ is a 3-separation. In [2], we described what reorderings of $(x_1, x_2, \ldots, x_n)$ produce another such sequence and specified what kinds of substructures of M result in these reorderings.

In this paper, we consider the behaviour of sequential 3-separations in M. In particular, our aim is to associate fixed ends with such a 3-separation so that we can use the results of [2]. Since we want this paper to include a description of sequential matroids that is as self-contained as possible, we shall state here a number of results from [2]. Let (A_1, B_1) be a sequential 3-separation. We call (A_1, B_1) *bisequential* if both $\mathrm{fcl}(A_1)$ and $\mathrm{fcl}(B_1)$ equal $E(M)$, and *unisequential* otherwise. In the latter case, suppose that $\mathrm{fcl}(A_1) = E(M)$. Then (A_1, B_1) is equivalent to $(A_1 - \mathrm{fcl}(B_1), \mathrm{fcl}(B_1))$ and, if $A = A_1 - \mathrm{fcl}(B_1)$, then, for every member (A_2, B_2) of the equivalence class $\mathcal{K}$ containing (A_1, B_1), we have $\{A_2 - \mathrm{fcl}(B_2), B_2 - \mathrm{fcl}(A_2)\} = \{A, \emptyset\}$. This gives us A as one fixed end for every member of $\mathcal{K}$. Our first task in treating the members of such an equivalence class $\mathcal{K}$ is to show that we can associate a second fixed end with the members of $\mathcal{K}$. Let S be a subset of $E(M)$ with $|S| \geq 3$. Then S is a *segment* if every 3-element subset of S is a triangle, and S is a *cosegment* if every 3-element subset of S is a triad. We call S a *fan* if there is an ordering $(s_1, s_2, \ldots, s_n)$ of the elements of S such that, for all $i \in \{1, 2, \ldots, n-2\}$,

(i) $\{s_i, s_{i+1}, s_{i+2}\}$ is either a triangle or a triad, and
(ii) when $\{s_i, s_{i+1}, s_{i+2}\}$ is a triangle, $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triad, and when $\{s_i, s_{i+1}, s_{i+2}\}$ is a triad, $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triangle.

This ordering $(s_1, s_2, \ldots, s_n)$ is called a *fan ordering* of S. When $n \geq 4$, the elements s_1 and s_n are the only elements of the fan that are not in both a triangle and a triad contained in S. We call these elements the *ends* of the fan S. The remaining elements of S are the *internal elements* of the fan. We denote the set of such elements by $I(S)$.

The second fixed end that one can associate with an equivalence class of unisequential 3-separations is obtained using the following result.

Theorem 1.1. *Let M be a 3-connected matroid with a 3-sequence $(X, x_1, x_2, \ldots, x_n)$ where $|X|$ and $n-1$ are both at least two and $E(M) - X$ is fully closed. Then $E(M) - X$ contains either*

(i) *a subset W such that, for every 3-sequence $(X, y_1, y_2, \ldots, y_n)$, the set W is the unique maximal subset of $E(M) - X$ that contains $\{y_{n-2}, y_{n-1}, y_n\}$ and is a segment, a cosegment, or a fan; or*