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Variants of the RSK algorithm adapted to combinatorial Macdonald polynomials ☆

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A B S T R A C T

We introduce variations of the Robinson–Schensted correspondence parametrized by positive integers p. Each variation gives a bijection between permutations and pairs of standard tableaux of the same shape. In addition to sharing many of the properties of the classical Schensted algorithm, the new algorithms are designed to be compatible with certain permutation statistics introduced by Haglund in the study of Macdonald polynomials. In particular, these algorithms provide an elementary bijective proof converting Haglund's combinatorial formula for Macdonald polynomials to an explicit combinatorial Schur expansion of Macdonald polynomials indexed by partitions μ satisfying $\mu_1 \leq 3$ and $\mu_2 \leq 2$. We challenge the research community to extend this RSK-based approach to more general classes of partitions.

This paper investigates and advocates an algorithmic approach to Macdonald's famous conjecture [19] requesting a combinatorial interpretation for the q, t-*Kostka matrix*, which gives the Schur expansion of modified Macdonald polynomials. The main idea is to invent

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variations of the Robinson–Schensted algorithm that can be used to translate Haglund's combinatorial formula for Macdonald polynomials [10] into the required Schur expansion. We assume the reader is familiar with general background on permutations, partitions, tableaux, and symmetric polynomials (as found, for instance, in [8, Part I], [18, Chap. 10], [20, Chap. I], [22, Chap. 3], and [23, Chap. 7]). However, no specific prior knowledge of Macdonald polynomials will be required. For brevity, we refer to the Robinson–Schensted bijection (which maps permutations to pairs of standard tableaux) as the *RSK algorithm*, although we will not need the general version of this algorithm due to Knuth (which maps matrices to pairs of semistandard tableaux).

The paper is structured as follows. Section 1 motivates our invention of new RSK algorithms by describing Macdonald polynomials, Schur polynomials, Macdonald's conjecture on the Schur expansion of Macdonald polynomials, prior progress on this conjecture, and our proposed algorithmic approach to this conjecture. Readers interested primarily in the combinatorics of RSK algorithms may safely omit this section. Section 2 reviews the ingredients of the classical Schensted algorithm on permutations—row insertion, tableau insertion, recording tableaux, and how to invert RSK. These elements are then generalized in Section 3 to define RSK algorithms parametrized by positive integers p. Section 4 proves that these new algorithms share many of the combinatorial properties of classical RSK; in particular, the recording tableaux for the new algorithms always agree with the classical recording tableau. Section 5 applies the parametrized RSK algorithms to give elementary bijective proofs of combinatorial formulas for the Schur expansions of Macdonald polynomials $\tilde{H}_\mu$ indexed by partitions μ satisfying $\mu_1 \leq 3$ and $\mu_2 \leq 2$.

1. Motivating problem: the Schur expansion of Macdonald polynomials

1.1. Haglund's combinatorial formula for Macdonald polynomials

Ian Macdonald introduced the symmetric polynomials now called *Macdonald polynomials* in 1988 [19]. There are several versions of Macdonald polynomials, each of which can be defined algebraically as the unique symmetric polynomials satisfying certain orthogonality or triangularity conditions. In this paper, we work with the *modified Macdonald polynomials*, denoted $\tilde{H}_\mu(X; q, t)$, indexed by integer partitions μ. In 2004, Haglund [10] discovered a celebrated combinatorial formula for the fundamental quasisymmetric expansion of these polynomials in terms of new permutation statistics inv_μ and maj_μ depending on μ. This formula was proved by Haglund, Haiman, and Loehr [11]. To minimize technical prerequisites for reading this paper, we will take Haglund's combinatorial formula as our *definition* of the modified Macdonald polynomials. See [11] for the original algebraic definitions of Macdonald polynomials and their connection to Haglund's formula.

Let $\mathrm{Par}(n)$ denote the set of all integer partitions of n, let $[n]$ denote the set $\{1, 2, \ldots, n\}$, and let S_n denote the set of permutations of $[n]$. Given a partition $\mu \in \mathrm{Par}(n)$, Haglund's formula has the form