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Courcelle's theorem for triangulations ☆

Benjamin A. Burton [a], Rodney G. Downey [b]

[a] School of Mathematics and Physics, The University of Queensland, Brisbane, QLD 4072, Australia
[b] School of Mathematics, Statistics and Operations Research, Victoria University, New Zealand

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In graph theory, Courcelle's theorem essentially states that, if an algorithmic problem can be formulated in monadic second-order logic, then it can be solved in linear time for graphs of bounded treewidth. We prove such a metatheorem for a general class of triangulations of arbitrary fixed dimension d, including all triangulated d-manifolds: if an algorithmic problem can be expressed in monadic second-order logic, then it can be solved in linear time for triangulations whose dual graphs have bounded treewidth.

We apply our results to 3-manifold topology, a setting with many difficult computational problems but very few parameterised complexity results, and where treewidth has practical relevance as a parameter. Using our metatheorem, we recover and generalise earlier fixed-parameter tractability results on taut angle structures and discrete Morse theory respectively, and prove a new fixed-parameter tractability result for computing the powerful but complex Turaev–Viro invariants on 3-manifolds.

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E-mail addresses: bab@maths.uq.edu.au (B.A. Burton), Rod.Downey@vuw.ac.nz (R.G. Downey).

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1. Introduction

Parameterised complexity is a relatively new and highly successful framework for understanding the computational complexity of "hard" problems for which we do not have a polynomial-time algorithm [18]. The key idea is to measure the complexity not just in terms of the input size (the traditional approach), but also in terms of additional *parameters* of the input or of the problem itself. The result is that, even if a problem is (for instance) NP-hard, we gain a richer theoretical understanding of those classes of inputs for which the problem is still tractable, and we acquire new practical tools for solving the problem in real software.

For example, finding a Hamiltonian cycle in an arbitrary graph is NP-complete, but for graphs of fixed treewidth $\leq k$ it can be solved in linear time in the input size [18]. In general, a problem is called *fixed-parameter tractable* in the parameter k if, for any class of inputs where k is universally bounded, the running time becomes *fixed degree* polynomial in the input size. More precisely, the setting will be languages $L \subseteq \Sigma^* \times \Sigma^*$[1] and L is fixed parameter tractable iff there is a fixed constant c, and an algorithm deciding "$(x, k) \in L$?" in time $f(k)(|x|^c)$. Here f is typically a computable function, as will be the case in the present paper. The standard examples are Vertex Cover (vertices cover edges) and Dominating Set (vertices cover vertices). On general graphs, the first is linear time ($c = 1$) fixed parameter tractable, and the second, the only known algorithm is to (essentially) check all possible k-element meaning the running time is $\Omega(|G|^k)$, behaviour considered as fixed parameter *intractability*. There is a completeness theory which places Dominating Set in its second level of a completeness hierarchy [16,17].

Treewidth in particular (which roughly measures how "tree-like" a graph is [38]) is extremely useful as a parameter. A great many graph problems are known to be fixed-parameter tractable in the treewidth, in a large part due to Courcelle's celebrated "metatheorem" [12,13]: for *any* decision problem P on graphs, if P can be framed using monadic second-order logic, then P can be solved in *linear time* for graphs of universally bounded treewidth $\leq k$.

The motivation behind this paper is to develop the tools of parameterised complexity for systematic use in the field of geometric topology, and in particular for 3-manifold topology. This is a field with natural and fundamental algorithmic problems, such as determining whether two knots or two triangulations are topologically equivalent, and in three dimensions such problems are often decidable but extremely complex [36].

Parameterised complexity is appealing as a theoretical framework for identifying when "hard" topological problems can be solved quickly. Unlike average-case complexity or generic complexity, it avoids the need to work with *random inputs*—something that still poses major difficulties for 3-manifold topology [20]. The viability of this framework is shown by recent parameterised complexity results in topological settings such as knot

[1] Think of (G, k) where G is a graph and k is some parameter.