Journal of Combinatorial Theory, Series A 146 (2017) 295–311

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Colorful versions of the Lebesgue, KKM, and Hex theorem

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ARTICLE INFO

Article history:
Received 12 February 2015
Available online xxxx

Keywords:
Lebesgue theorem
KKM theorem
Simple polytopes
Quasitoric manifolds
Cup-length

ABSTRACT

Following and developing ideas of R. Karasev (2014) [10], we extend the Lebesgue theorem (on covers of cubes) and the Knaster–Kuratowski–Mazurkiewicz theorem (on covers of simplices) to different classes of convex polytopes (colored in the sense of M. Joswig). We also show that the n-dimensional Hex theorem admits a generalization where the n-dimensional cube is replaced by a n-colorable simple polytope. The use of specially designed *quasitoric manifolds*, with easily computable cohomology rings and the cohomological cup-length, offers a great flexibility and versatility in applying the general method.

1. Introduction

The well known connection between the classical Lyusternik–Schnirelmann category (LS-category) and the cohomological cup-length is a simple, yet elegant and powerful method of studying geometric/topological properties of a space by computable invariants arising in algebraic topology. Together with its generalizations and ramifications, this connection is indeed one of evergreen themes of geometry and topology.

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http://dx.doi.org/10.1016/j.jcta.2016.10.002

It was an interesting recent observation of Karasev [10] that a similar cohomological cup-length approach can be utilized for the proof of some results of more combinatorial nature, including the following two classical results of Lebesgue, and Knaster, Kuratowski, Mazurkiewicz (KKM).

Theorem 1.1 *(*Lebesgue*). If the unit cube $[0,1]^n$ is covered by a finite family $\{X_i\}_{i\in I}$ of closed sets so that no point is included in more than n sets, then one of them must intersect two opposite facets of the cube.*

Theorem 1.2 *(*KKM*). If a non-degenerate simplex $\Delta^n \subset \mathbb{R}^n$ is covered by a finite family $\{F_i\}_{i\in I}$ of closed sets so that no point is covered more than n times then one of the sets F_i intersects all the facets of Δ^n.*

The method of Karasev was based on the use of cohomological properties of (both non-singular and singular) toric varieties. In particular, he was able to unify Theorems 1.1 and 1.2 and interpret them as special cases of a single statement valid for all simple polytopes.

Theorem 1.3. [10, Theorem 5.2.] *Suppose that a simple polytope $P \subset \mathbb{R}^n$ is covered by a family of closed sets $\{X_i\}_{i\in I}$ with covering multiplicity at most n. Then for some $i \in I$ the set X_i intersects at least $n+1$ facets of P.*

We continue this study by methods of toric topology, emphasizing the role of quasitoric manifolds [7,4]. We focus on special classes of simple polytopes including the class of n-colorable simple polytopes which were introduced by Joswig in [9]. The associated classes of quasitoric manifolds have computable and often favorable cohomological properties, which have already found applications outside toric topology [2,3].

Our central results, the 'Colorful Lebesgue theorem' (Theorem 3.1) and the 'Colorful KKM-theorem' (Theorem 4.1), together with their companions Theorem 3.2 and Theorem 4.2, are designed to include Theorems 1.1 and 1.2 as special cases and to illuminate the role of special classes of quasitoric manifolds over n-colorable and $(n+1)$-colorable simple polytopes.

In the same vein we prove the 'Colorful Hex theorem' Theorem 6.1 and describe a 'Colorful Voronoi-Hex game' played by n players on an n-dimensional Voronoi checkerboard.

We refer the reader, curious or intrigued by the use of the word 'colorful' in these statements, to [1] and [14] for a sample of results illustrating how the term 'colorful' gradually acquired (almost) a technical meaning in many areas of geometric and topological combinatorics.

2. Overview and preliminaries

A basic insight from the theory of *Lebesgue covering dimension* is that an n-dimensional space cannot be covered by a family $\mathcal{U}$ of open sets which are *'small in size'* unless