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The Selberg integral and Young books

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ABSTRACT

The Selberg integral is an important integral first evaluated by Selberg in 1944. Stanley found a combinatorial interpretation of the Selberg integral in terms of permutations. In this paper, new combinatorial objects "Young books" are introduced and shown to have a connection with the Selberg integral. This connection gives an enumeration formula for Young books. It is shown that special cases of Young books become standard Young tableaux of various shapes: shifted staircases, squares, certain skew shapes, and certain truncated shapes. As a consequence, product formulas for the number of standard Young tableaux of these shapes are obtained.

1. Introduction

The Selberg integral is the following integral first evaluated by Selberg [8] in 1944:

$$S_n(\alpha,\beta,\gamma) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} x_i^{\alpha-1}(1-x_i)^{\beta-1} \prod_{1\le i<j\le n} |x_i - x_j|^{2\gamma} dx_1 \cdots dx_n \qquad (1)$$

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$$= \prod_{j=1}^{n} \frac{\Gamma(\alpha + (j-1)\gamma)\Gamma(\beta + (j-1)\gamma)\Gamma(1 + j\gamma)}{\Gamma(\alpha + \beta + (n + j - 2)\gamma)\Gamma(1 + \gamma)},$$

where n is a positive integer and α, β, γ are complex numbers such that $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, and $\mathrm{Re}(\gamma) > -\min\{1/n, \mathrm{Re}(\alpha)/(n-1), \mathrm{Re}(\beta)/(n-1)\}$. We refer the reader to Forrester and Warnaar's exposition [3] for the history and importance of the Selberg integral.

In [10, Exercise 1.11 (b)] Stanley gives a combinatorial interpretation of the Selberg integral when the exponents $\alpha - 1, \beta - 1$ and 2γ are nonnegative integers by introducing certain permutations. In this paper we define "Selberg books" which are essentially a graphical representation of these permutations as fillings of certain Young diagrams. We then define "Young books" which are special Selberg books. Young books are a generalization of both shifted Young tableaux of staircase shape and standard Young tableaux of square shape. We show that there is a simple relation between the number of Selberg books and the number of Young books by finding generating functions for both objects.

It is well known that the number of standard Young tableaux has a nice product formula due to Frame, Robinson, and Thrall [4] in which every factor is at most the size of the shape. However, the number of standard Young tableaux of a skew shape or a truncated shape does not have such a product formula since it contains a large prime factor compared to the size of the shape. A truncated shape is a diagram obtained from a usual Young diagram in English convention by removing cells from its southwest corner. Standard Young tableaux of truncated shapes were recently considered by Adin and Roichman [2]. They showed that the number of geodesics between two antipodes in the flip graph of triangle-free triangulations is equal to twice the number of standard Young tableaux of certain shifted truncated shape. Adin, King, and Roichman [1], Sun [11–13] and Panova [6] showed that the number of standard Young tableaux of certain truncated shapes has a product formula. As a consequence of our formula for the Young books, we obtain some product formulas for the number of standard Young tableaux of some skew shapes and truncated shapes.

We note that Sun [11,12] also found a connection between the Selberg integral and the number of standard Young tableaux of certain shape with a somewhat similar but different approach. His idea is, roughly speaking, to write the number of standard Young tableaux of a certain shape as an integral, rewrite the integral as a determinant, and evaluate the determinant to get the Selberg integral.

This paper is organized as follows. In Section 2 we review Stanley's combinatorial interpretation of the Selberg integral. In Section 3 we define Selberg books and Young books in a simple form, related to the Selberg integral when $\alpha = \beta = 1$. We use generating functions to show that there is a simple relation between their cardinalities. Using this relation and (1) we obtain a formula for the number of Young books. In Section 4 we