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A generalisation of a second partition theorem of Andrews to overpartitions

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A R T I C L E I N F O

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A B S T R A C T

In 1968 and 1969, Andrews proved two partition theorems of the Rogers–Ramanujan type which generalise Schur's celebrated partition identity (1926). Andrews' two generalisations of Schur's theorem went on to become two of the most influential results in the theory of partitions, finding applications in combinatorics, representation theory and quantum algebra. In a recent paper, the author generalised the first of these theorems to overpartitions, using a new technique which consists in going back and forth between q-difference equations on generating functions and recurrence equations on their coefficients. Here, using a similar method, we generalise the second theorem of Andrews to overpartitions.

1. Introduction

A partition of n is a non-increasing sequence of natural numbers whose sum is n. An overpartition of n is a partition of n in which the first occurrence of a number may be

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overlined. For example, there are 14 overpartitions of 4: 4, $\bar{4}$, $3+1$, $\bar{3}+1$, $3+\bar{1}$, $\bar{3}+\bar{1}$, $2+2$, $\bar{2}+2$, $2+1+1$, $\bar{2}+1+1$, $2+\bar{1}+1$, $\bar{2}+\bar{1}+1$, $1+1+1+1$ and $\bar{1}+1+1+1$.

In 1926, Schur [19] proved the following partition identity.

Theorem 1.1 *(Schur). Let n be a positive integer. Let $D_1(n)$ denote the number of partitions of n into distinct parts congruent to 1 or 2 modulo 3. Let $E_1(n)$ denote the number of partitions of n of the form $n = \lambda_1 + \cdots + \lambda_s$ where $\lambda_i - \lambda_{i+1} \geq 3$ with strict inequality if $\lambda_{i+1} \equiv 0 \mod 3$. Then $D_1(n) = E_1(n)$.*

For example, for $n = 9$, the partitions counted by $D_1(9)$ are $8+1$, $7+2$ and $5+4$ and the partitions counted by $E_1(9)$ are 9, $8+1$ and $7+2$. Thus $D_1(9) = E_1(9) = 3$.

Several proofs of Schur's theorem have been given using a variety of different techniques such as bijective mappings [9,10], the method of weighted words [2], and recurrences [3,5,7].

Schur's theorem was subsequently generalised to overpartitions by Lovejoy [16] using the method of weighted words. The case $k = 0$ corresponds to Schur's theorem.

Theorem 1.2 *(Lovejoy). Let $D_1(k,n)$ denote the number of overpartitions of n into parts congruent to 1 or 2 modulo 3 with k non-overlined parts. Let $E_1(k,n)$ denote the number of overpartitions of n with k non-overlined parts, where parts differ by at least 3 if the smaller is overlined or both parts are divisible by 3, and parts differ by at least 6 if the smaller is overlined and both parts are divisible by 3. Then $D_1(k,n) = E_1(k,n)$.*

Theorem 1.2 was then proved bijectively by Raghavendra and Padmavathamma [18], and using q-difference equations and recurrences by the author [14].

Andrews extended the ideas of his proofs of Schur's theorem to prove two much more general theorems on partitions with difference conditions [4,6]. But before stating these results in their full generality we need to introduce some notation. Let $A = \{a(1), ..., a(r)\}$ be a set of r distinct positive integers such that $\sum_{i=1}^{k-1} a(i) < a(k)$ for all $1 \leq k \leq r$ and the $2^r - 1$ possible sums of distinct elements of A are all distinct. We denote this set of sums by $A' = \{\alpha(1), ..., \alpha(2^r - 1)\}$, where $\alpha(1) < \cdots < \alpha(2^r - 1)$. Let us notice that $\alpha(2^k) = a(k+1)$ for all $0 \leq k \leq r-1$ and that any α between $a(k)$ and $a(k+1)$ has largest summand $a(k)$. Let N be a positive integer with $N \geq \alpha(2^r - 1) = a(1) + \cdots + a(r)$. We further define $\alpha(2^r) = a(r+1) = N + a(1)$. Let A_N denote the set of positive integers congruent to some $a(i) \mod N$, $-A_N$ the set of positive integers congruent to some $-a(i) \mod N$, A'_N the set of positive integers congruent to some $\alpha(i) \mod N$ and $-A'_N$ the set of positive integers congruent to some $-\alpha(i) \mod N$. Let $\beta_N(m)$ be the least positive residue of $m \mod N$. If $\alpha \in A'$, let $w(\alpha)$ be the number of terms appearing in the defining sum of α and $v(\alpha)$ the smallest $a(i)$ appearing in this sum.

To illustrate these notations in the remainder of this paper, it might be useful to consider the example where $a(k) = 2^{k-1}$ for $1 \leq k \leq r$ and $\alpha(k) = k$ for $1 \leq k \leq 2^r - 1$.

We are now able to state Andrews' generalisations of Schur's theorem.