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The greedy basis equals the theta basis: A rank two haiku

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Dedicated to the memory of Andrei Zelevinsky

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ABSTRACT

We prove the equality of two canonical bases of a rank 2 cluster algebra, the greedy basis of Lee–Li–Zelevinsky and the theta basis of Gross–Hacking–Keel–Kontsevich.

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1. Introduction

Cluster algebras are commutative rings with partial bases of a special form, originally discovered in the context of dual canonical bases in Lie theory [6]. Their axiomatics

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encapsulates the fact that many kinds of canonical bases in nature have large subsets which are governed by a uniform combinatorics. Elements of these subsets are monomials in distinguished elements called cluster variables, which are grouped into overlapping collections called clusters. Each cluster has an associated skew-symmetrizable matrix and the entire cluster algebra can be reconstructed recursively from any particular cluster along with this matrix.

A fundamental issue in the theory is understanding natural completions of the partial basis of cluster monomials to a full basis of the cluster algebra. Depending on the context, this question can be analyzed from a wide range of perspectives drawn from representation theory, geometry, combinatorics, and mathematical physics [4,16,5,22,2,26,8]. In general, one expects any cluster algebra to admit several natural bases related in potentially subtle ways. A basic example of this is the relationship between the dual canonical and dual semicanonical bases of the coordinate ring of the positive unipotent subgroup of a simple algebraic group [9]. This example also illustrates that, in general, even determining whether or not two constructions of canonical bases in a cluster algebra lead to the same result is nontrivial. The purpose of the present paper is to compare two such constructions for cluster algebras associated to 2×2 skew-symmetrizable matrices.

The first basis we consider is the greedy basis of [27,18]. Every cluster algebra is contained in the ring of Laurent polynomials in the cluster variables of any of its clusters. The recently-confirmed positivity conjecture, proved in the rank 2 case in [20,25] and in the general case in [21,12], asserts that the coefficients of the Laurent expansion of any cluster variable are positive integers. The greedy basis is defined so that all of its elements, not just cluster variables, have positive Laurent expansions in any cluster and that the coefficients of any such Laurent expansion are as small as possible. The resulting coefficients turn out to enumerate combinatorial objects called compatible pairs related to maximal Dyck paths.

The second basis we consider is the theta basis of [12]. Unlike the greedy basis it is defined for cluster algebras of arbitrary rank. In fact, this basis is a special case of a much more general construction based on two concepts. The first is that of scattering diagram introduced in [17] in two dimensions and in [14] in all dimensions. This diagram encodes the relations among cluster transformations and also among elements of the tropical vertex group. The second is a combinatorial notion of broken line, introduced in [10] with their theory further developed in [3] and then [11]. The coefficients of Laurent expansions of theta basis elements enumerate broken lines. These are piecewise-linear paths in a tropicalization of the cluster variety whose points of non-linearity lie along the scattering diagram. Morally broken lines capture the geometry of holomorphic disks in the mirror cluster variety.

Our main result is the following.

Theorem 1.1. *Let $\mathcal{A}$ be a rank 2 cluster algebra. The greedy and theta bases of $\mathcal{A}$ coincide.*

The proof is based on an analysis of exactly which monomials may appear in elements of the theta basis. It can be shown that elements of the greedy basis are essentially