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Four-dimensional polytopes of minimum positive semidefinite rank ☆

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ABSTRACT

The positive semidefinite (psd) rank of a polytope is the size of the smallest psd cone that admits an affine slice that projects linearly onto the polytope. The psd rank of a d-polytope is at least $d + 1$, and when equality holds we say that the polytope is psd-minimal. In this paper we develop new tools for the study of psd-minimality and use them to give a complete classification of psd-minimal 4-polytopes. The main tools introduced are trinomial obstructions, a new algebraic obstruction for psd-minimality, and the slack ideal of a polytope, which encodes the space of realizations of a polytope up to projective equivalence.

Our central result is that there are 31 combinatorial classes of psd-minimal 4-polytopes. We provide combinatorial information and an explicit psd-minimal realization in each class. For 11 of these classes, every polytope in them is psd-minimal, and these are precisely the combinatorial classes

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of the known projectively unique 4-polytopes. We give a complete characterization of psd-minimality in the remaining classes, encountering in the process counterexamples to some open conjectures.

1. Introduction

The *positive semidefinite (psd) rank* of a convex set was introduced in [12], and it can be seen as a measure of geometric complexity of that set. Let $\mathcal{S}^k$ denote the vector space of all real symmetric $k \times k$ matrices with the inner product $\langle A, B \rangle := \mathrm{trace}(AB)$, and let $\mathcal{S}^k_+$ be the cone of positive semidefinite matrices in $\mathcal{S}^k$. A polytope $P \subset \mathbb{R}^d$ is said to have a *psd lift* of size k if there is an affine space $L \subset \mathcal{S}^k$ and a linear map $\pi : \mathcal{S}^k \to \mathbb{R}^d$ such that $P = \pi(\mathcal{S}^k_+ \cap L)$. The psd rank of P, $\mathrm{rank}_{\mathrm{psd}}(P)$, is the smallest k such that P has a psd lift of size k. Linear optimization over a polytope can be achieved via semidefinite programming over its psd lift. Thus a lift of small size (alternatively, small psd rank of the polytope) implies, in principle, the possibility of efficiently solving a linear optimization problem over this polytope. These features have attracted much research on the psd rank of polytopes in recent years, with several exciting new results coming from optimization and computer science [1,7,8,18–20]. For a survey on psd rank of nonnegative matrices, see [6].

Since polytopes of small psd rank can admit efficient algorithms for linear optimization, there is much incentive to understand them. If P is a d-polytope, then $\mathrm{rank}_{\mathrm{psd}}(P) \geq d+1$, and if equality holds, we say that P is *psd-minimal*. These polytopes are a natural place to start the study of small psd rank, a task that was initiated in [13]. A well-known example of a psd-minimal polytope is the *stable set polytope* of a *perfect graph* [17]. In this case, psd-minimality implies that the size of a largest stable set in a perfect graph can be found in polynomial time, while computing the size of a largest stable set in a graph is NP-hard in general. The existence of a small psd lift for the stable set polytopes of perfect graphs is the only known proof of the polynomial time solvability of the stable set problem in this class of graphs.

The stable set polytope of a perfect graph is also an example of a *2-level polytope* [11]. These are polytopes with the property that for each facet of the polytope there is a unique parallel translate of its affine hull containing all vertices of the polytope that are not on the facet. All 2-level polytopes are psd-minimal and affinely equivalent to 0/1-polytopes with additional special properties, yet they are far from well-understood and offer many challenges. Several groups of researchers have been studying them recently [2,3,15]. The study of psd-minimal polytopes is an even more ambitious task than that of 2-level polytopes, yet it is an important step in understanding the phenomenon of small psd rank, and offer a rich set of examples for honing psd rank techniques.