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A bijection for rooted maps on general surfaces

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A B S T R A C T

We extend the Marcus–Schaeffer bijection between orientable rooted bipartite quadrangulations (equivalently: rooted maps) and orientable labeled one-face maps to the case of *all* surfaces, that is orientable and non-orientable as well. This general construction requires new ideas and is more delicate than the special orientable case, but it carries the same information. In particular, it leads to a uniform combinatorial interpretation of the counting exponent $\frac{5(h-1)}{2}$ for both orientable and non-orientable rooted connected maps of Euler characteristic $2-2h$, and of the algebraicity of their generating functions, similar to the one previously obtained in the orientable case via the Marcus–Schaeffer bijection. It also shows that the renormalization factor $n^{1/4}$ for distances between vertices is universal for maps on all surfaces: the renormalized profile and radius in a uniform random pointed bipartite quadrangulation on any fixed surface converge in distribution when the size n tends to infinity. Finally, we extend the Miermont and Ambjørn–Budd bijections to the general setting of all surfaces. Our construction opens the

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way to the study of Brownian surfaces for any compact 2-dimensional manifold.

1. Introduction

1.1. Maps

Maps (a.k.a. ribbon graphs, or embedded graphs) are combinatorial structures that describe the embedding of a graph in a surface (see Section 2 for precise definitions). These objects have received much attention from many different viewpoints, because of their deep connections with various branches of discrete mathematics, algebra, or physics (see e.g. [21,28] and references therein). In particular, maps have remarkable enumerative properties, and the enumeration of maps (either by generating functions, matrix integral techniques, algebraic combinatorics, or bijective methods) is now a well established domain on its own. The reader may consult [1,5,7,15] for entry points into this fast-growing literature. This paper is devoted to the extension of the bijective method of map enumeration to the case of *all* surfaces (orientable and non-orientable), and to its first consequences in terms of combinatorial enumeration and probabilistic results.

1.2. Orientable surfaces

Let us first recall briefly the situation in the orientable case. A fundamental result of Bender and Canfield [3], obtained with generating functions, says that the number $m_g(n)$ of rooted maps with n edges on the orientable surface of genus $g \geq 0$ (obtained by adding g handles to a sphere, see Section 2) is asymptotically equivalent to

$$m_g(n) \sim t_g n^{\frac{5(g-1)}{2}} 12^n, \quad n \to \infty, \tag{1}$$

for some $t_g > 0$. In the planar case ($g = 0$) this follows from the exact formula $m_0(n) = \frac{2 \cdot 3^n (2n)!}{(n+2)!n!}$ due to Tutte [38], whose combinatorial interpretation was given by Cori–Vauquelin [18] and much improved by Schaeffer [37]. The bijective enumerative theory of planar maps has since grown into a domain of research of its own, out of the scope of this introduction; consult [1,7] and references therein. For general g, the combinatorial interpretation of Formula (1), and, in particular, of the counting exponent $\frac{5(g-1)}{2}$ was given in [16], using an extension of the Cori–Vauquelin–Schaeffer bijection to the case of higher genus orientable surfaces previously given by Marcus and Schaeffer in [31]. The bijection of [16,31] associates maps on a surface with labeled one-face maps on the same surface. The latter have a much simpler combinatorial structure, and can be enumerated by elementary ways. Moreover, the bijective toolbox proved to be relatively flexible, and enabled to prove formulas similar to (1) for many different families of orientable maps [14], extending results previously obtained by generating functions [22].