Journal of Combinatorial Theory, Series A 145 (2017) 308–339

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Sign variation, the Grassmannian, and total positivity

Steven N. Karp [1]

Department of Mathematics, University of California, Berkeley, United States

ARTICLE INFO

Article history:
Received 1 April 2015
Available online 31 August 2016

Keywords:
Sign variation
Total positivity
Totally nonnegative Grassmannian
Amplituhedron
Grassmann polytope
Positroid
Oriented matroid

ABSTRACT

The *totally nonnegative Grassmannian* is the set of k-dimensional subspaces V of $\mathbb{R}^n$ whose nonzero Plücker coordinates all have the same sign. Gantmakher and Krein (1950) and Schoenberg and Whitney (1951) independently showed that V is totally nonnegative iff every vector in V, when viewed as a sequence of n numbers and ignoring any zeros, changes sign at most $k-1$ times. We generalize this result from the totally nonnegative Grassmannian to the entire Grassmannian, showing that if V is *generic* (i.e. has no zero Plücker coordinates), then the vectors in V change sign at most m times iff certain sequences of Plücker coordinates of V change sign at most $m-k+1$ times. We also give an algorithm which, given a non-generic V whose vectors change sign at most m times, perturbs V into a generic subspace whose vectors also change sign at most m times. We deduce that among all V whose vectors change sign at most m times, the generic subspaces are dense. These results generalize to oriented matroids. As an application of our results, we characterize when a generalized *amplituhedron* construction, in the sense of Arkani-Hamed and Trnka (2013), is well defined. We also give two ways of obtaining the *positroid*

E-mail address: skarp@berkeley.edu.
URL: http://math.berkeley.edu/~skarp/.

[1] This work was supported by a Chateaubriand fellowship, an NSERC postgraduate scholarship, and NSF grant DMS-1049513.

http://dx.doi.org/10.1016/j.jcta.2016.08.003

cell of each V in the totally nonnegative Grassmannian from the sign patterns of vectors in V.

1. Introduction and main results

The *(real) Grassmannian* $\mathrm{Gr}_{k,n}$ is the set of k-dimensional subspaces of $\mathbb{R}^n$. Given $V \in \mathrm{Gr}_{k,n}$, take a $k \times n$ matrix X whose rows span V; then for k-subsets $I \subseteq \{1, \cdots, n\}$, we let $\Delta_I(V)$ be the $k \times k$ minor of X restricted to the columns in I, called a *Plücker coordinate*. (The $\Delta_I(V)$ depend on our choice of X only up to a global constant.) If all nonzero $\Delta_I(V)$ have the same sign, then V is called *totally nonnegative*, and if in addition no $\Delta_I(V)$ equals zero, then V is called *totally positive*. For example, the span V of $(1, 0, 0, -1)$ and $(-1, 2, 1, 3)$ is a totally nonnegative element of $\mathrm{Gr}_{2,4}$, but V is not totally positive since $\Delta_{\{2,3\}}(V) = 0$.

The set $\mathrm{Gr}_{k,n}^{\geq 0}$ of totally nonnegative $V \in \mathrm{Gr}_{k,n}$, called the *totally nonnegative Grassmannian*, has become a hot topic in algebraic combinatorics in the past two decades. The general algebraic study of total positivity for split reductive connected algebraic groups G over $\mathbb{R}$, and partial flag varieties G/P, was initiated by Lusztig [14], of which $\mathrm{Gr}_{k,n}^{\geq 0}$ corresponds to the special case $G/P = \mathrm{Gr}_{k,n}$. Of particular interest is the stratification of $\mathrm{Gr}_{k,n}^{\geq 0}$ according to whether each Δ_I is zero or nonzero. This stratification is a cell decomposition, which was conjectured by Lusztig [14] and proved by Rietsch [19] (for the general case G/P), and later understood combinatorially by Postnikov [18].

This general theory traces its origin to the study of *totally positive matrices* in the 1930s, in the context of oscillation theory in analysis. Here positivity conditions on matrices can imply special oscillation and spectral properties. A well-known result of this kind is due to Gantmakher and Krein [9], which states that if an $n \times n$ matrix X is *totally positive* (i.e. all $\binom{2n}{n}$ minors of X are positive), then the n eigenvalues of X are distinct positive reals. Gantmakher and Krein [8] also gave a characterization of (what would later be called) the totally nonnegative and totally positive Grassmannians in terms of sign variation. To state their result, we introduce some notation. For $v \in \mathbb{R}^n$, let $\mathrm{var}(v)$ be the number of times v (viewed as a sequence of n numbers, ignoring any zeros) changes sign, and let

$$\overline{\mathrm{var}}(v) := \max\{\mathrm{var}(w) : w \in \mathbb{R}^n \text{ such that } w_i = v_i \text{ for all } 1 \leq i \leq n \text{ with } v_i \neq 0\}.$$

(We use the convention $\mathrm{var}(0) := -1$.) For example, if $v := (1, -1, 0, -2) \in \mathbb{R}^4$, then $\mathrm{var}(v) = 1$ and $\overline{\mathrm{var}}(v) = 3$.

Theorem 1.1 *(Chapter V, Theorems 3 and 1 of [8]).*
(i) $V \in \mathrm{Gr}_{k,n}$ *is totally nonnegative iff* $\mathrm{var}(v) \leq k - 1$ *for all* $v \in V$.
(ii) $V \in \mathrm{Gr}_{k,n}$ *is totally positive iff* $\overline{\mathrm{var}}(v) \leq k - 1$ *for all* $v \in V \setminus \{0\}$.