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High girth augmented trees are huge

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ABSTRACT

Let G be a graph consisting of a complete binary tree of depth h together with one back edge leading from each leaf to one of its ancestors, and suppose that the girth of G exceeds g. Let $h = h(g)$ be the minimum possible depth of such a graph. The existence of such graphs, for arbitrarily large g, is proved in [2], where it is shown that $h(g)$ is at most some version of the Ackermann function. Here we show that this is tight and the growth of $h(g)$ is indeed Ackermannian.

1. Introduction

An r-augmented tree is a graph consisting of a rooted tree (called the underlying tree) plus edges from each leaf to r of its ancestors (called here back edges). A complete d-ary tree of height m is a rooted tree whose internal vertices have d children and whose leaves have distance m from the root. For positive integers d, r, g let a (d, r, g)-graph be an r-augmented complete d-ary tree with girth exceeding g.

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These graphs were introduced in [2], where it is proved that they exist for all integers d, r and g. As shown in [2] these graphs provide simple explicit constructions of graphs and hypergraphs of high girth and high chromatic number, and are also useful in the investigation of several natural list coloring problems. In particular they provide sparse non-k-choosable bipartite graphs, showing that maximum average degree at most $2(k-1)$ is a sharp sufficient condition for k-choosability in bipartite graphs, even when requiring large girth. See [2] for more details. The graphs constructed in that paper are huge, their number of vertices grow like some version of the Ackermann function. It is also proved in [2] that the number of vertices must be large, specifically, the number of vertices of any $(2, 1, g)$-graph is at least a tower of height $2^{\Omega(g)}$. Although that's a fast growing function of g, the upper bound is far, far larger. Here we show that the upper bound is closer to the correct behavior, and the size of any $(2, 1, g)$ graph must indeed be Ackermannian.

For a function f, define its iterates f_i by putting $f_0(x) = x$, $f_1(x) = f(x)$ and more generally $f_i(x) = f(f_{i-1}(x))$ for all $i \geq 1$. Thus, for example, if $f(x) = 2^x$ then $f_i(1)$ is the tower of exponentials $Tower(i)$ and $Tower_i(1)$ is the iterated Tower, usually denoted by $WOW(i)$.

For positive integers x, r, g, a $TB(x, r, g)$-graph (where TB stands for Tree-Based) is a graph G whose vertices are all vertices of a complete binary tree T of some depth, called the underlying tree of G. Note that the graph G does not contain any of the tree edges, it is simply convenient to represent its vertices by the vertices of the underlying tree. The vertices are partitioned into levels, according to their distance from the leaves in the tree T. The leaves are at level 0, their neighbors (in T) at level 1, and so on. The edges of G must satisfy the following property. There is some integer $x' \geq x$ and a set L of leaves of the underlying tree, consisting of at least a fraction of $2^{-x'/4}$ of all leaves, so that each leaf in L has at least r back edges, all leading to levels higher than x'. Finally, the girth of G is bigger than g. Note that G is bipartite, all its edges are incident with the leaves of its underlying tree, and thus for $r = 1$ such a graph is simply a union of vertex disjoint stars.

For, say, $x \geq 100$ and integers $r \geq 2, g \geq 4$, let $m(x, r, g)$ denote the largest integer m such that any $TB(x, r, g)$-graph contains a set of at least a fraction of $2^{-m/8}$ of the leaves, so that each member of this set has a back edge leading to a level higher than m.

Note that by definition, the function $m(x, r, g)$ is monotone increasing in all its 3 variables.

It is not difficult to prove that for any $g > 101$ the minimum possible height of a $(2, 1, g)$-graph is at least $m(100, 2, g/2)$ (see Subsection 3.1 for the argument). Therefore, in order to establish the Ackermannian behavior of the size of any $(2, 1, g)$-graph it suffices to prove such a lower bound for $m(100, 2, g)$. This is done in the next sections.

In Section 2 we show that $m(100, r, 4)$ grows at least like a tower function of r and that $m(100, r, 8)$ grows at least like a WOW function of r. In Section 3 we note that a similar reasoning implies that whenever g is increased by a factor of 2, the lower bound for the growth of $m(100, r, g)$ as a function of r shifts to the next level in the Ackermann hierarchy. In Section 4 we establish an upper bound for $m(x, r, g)$ (and in fact for a