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Square lattice walks avoiding a quadrant

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ABSTRACT

In the past decade, a lot of attention has been devoted to the enumeration of walks with prescribed steps confined to a convex cone. In two dimensions, this means counting walks in the first quadrant of the plane (possibly after a linear transformation).

But what about walks in non-convex cones? We investigate the two most natural cases: first, square lattice walks avoiding the negative quadrant $\mathcal{Q}_1 = \{(i,j) : i < 0 \text{ and } j < 0\}$, and then, square lattice walks avoiding the West quadrant $\mathcal{Q}_2 = \{(i,j) : i < j \text{ and } i < -j\}$. In both cases, the generating function that counts walks starting from the origin is found to differ from a simple D-finite series by an algebraic one. We also obtain closed form expressions for the number of n-step walks ending at certain prescribed endpoints, as a sum of three hypergeometric terms.

One of these terms already appears in the enumeration of square lattice walks confined to the cone $\{(i,j) : i + j \geq 0 \text{ and } j \geq 0\}$, known as Gessel's walks. In fact, the enumeration of Gessel's walks follows, by the reflection principle, from the enumeration of walks starting from $(-1,0)$ and avoiding $\mathcal{Q}_1$. Their generating function turns out to be purely algebraic (as the generating function of Gessel's walks).

Another approach to Gessel's walks consists in counting walks that start from $(-1,1)$ and avoid the West quadrant $\mathcal{Q}_2$. The associated generating function is D-finite but transcendental.

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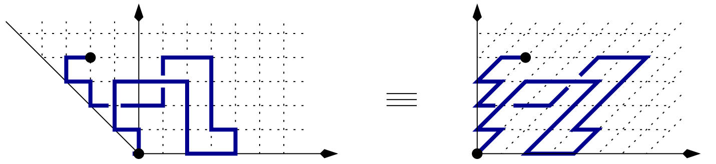

Fig. 1. Square lattice walks staying in a 135° wedge are equivalent to quadrant walks with steps $\rightarrow, \nearrow, \leftarrow, \swarrow$.

1. Introduction

In recent years, the enumeration of lattice walks confined to convex cones has attracted a lot of attention. In two dimensions, this means counting walks in the intersection of two half-spaces, which we can always assume (Fig. 1) to form the first quadrant $\mathcal{Q} = \{(i,j) : i \geq 0 \text{ and } j \geq 0\}$. The problem is then completely specified by prescribing a starting point and a set of allowed steps. The two most natural examples are walks on the square lattice (with steps $\rightarrow, \uparrow, \leftarrow, \downarrow$), and walks on the *diagonal* square lattice (with steps $\nearrow, \nwarrow, \swarrow, \searrow$). Both cases can be solved via the classical reflection principle [15,16]. The enumeration usually records the length n of the walk (with a variable t), and the coordinates (i,j) of its endpoint (with variables x and y). For instance, the generating function of square lattice walks starting from $(0,0)$ and confined to $\mathcal{Q}$ is [16,10]:

$$Q(x,y) = \sum_{i,j,n \geq 0} \frac{(i+1)(j+1)}{(n+1)(n+2)} \binom{n+2}{\frac{n-i-j}{2}} \binom{n+2}{\frac{n+i-j+2}{2}} x^i y^j t^n, \tag{1}$$

where the sum is restricted to integers i, j, n such that n and $i+j$ have the same parity. (To lighten notation, we ignore the dependence in t of this series.) This series is *D-finite* [21]: this means that it satisfies a linear differential equation in each of its variables t, x and y, with coefficients in the field $\mathbb{Q}(t,x,y)$ of rational functions in t, x and y.

In the past decade, a systematic study of quadrant walks with *small steps* (that is, steps in $\{-1,0,1\}^2$) has been carried out, and a complete classification is now available. For walks starting at $(0,0)$, the generating function is D-finite if and only if a certain group of birational transformations is finite. The proof combines an attractive combination of approaches: algebraic [7,10,14,15,23,26], computer-algebraic [3,18,19], analytic [4,20,28], asymptotic [5,11,22,24].

The most intriguing D-finite case is probably Gessel's model, illustrated in Fig. 1. Around 2000, Ira Gessel conjectured that the number of $2n$-step walks of this type starting and ending at $(0,0)$ was

$$g_{0,0}(2n) = 16^n \frac{(1/2)_n (5/6)_n}{(2)_n (5/3)_n}, \tag{2}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the ascending factorial. A computer-aided proof of this conjecture was finally found in 2009 by Kauers, Koutschan and Zeilberger [18]. A year