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# Invitation to intersection problems for finite sets

Peter Frankl [a], Norihide Tokushige [b,1]

[a] *Alfréd Rényi Institute of Mathematics, H-1364 Budapest, P.O. Box 127, Hungary*
[b] *College of Education, Ryukyu University, Nishihara, Okinawa 903-0213, Japan*

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A B S T R A C T

Extremal set theory is dealing with families, $\mathcal{F}$ of subsets of an $n$-element set. The usual problem is to determine or estimate the maximum possible size of $\mathcal{F}$, supposing that $\mathcal{F}$ satisfies certain constraints. To limit the scope of this survey most of the constraints considered are of the following type: any $r$ subsets in $\mathcal{F}$ have at least $t$ elements in common, all the sizes of pairwise intersections belong to a fixed set, $L$ of natural numbers, there are no $s$ pairwise disjoint subsets. Although many of these problems have a long history, their complete solutions remain elusive and pose a challenge to the interested reader.

Most of the paper is devoted to sets, however certain extensions to other structures, in particular to vector spaces, integer sequences and permutations are mentioned as well. The last part of the paper gives a short glimpse of one of the very recent developments, the use of semidefinite programming to provide good upper bounds.



## Contents



*E-mail addresses:* peter.frankl@gmail.com (P. Frankl), hide@edu.u-ryukyu.ac.jp (N. Tokushige).

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## 1. Introduction

For a positive integer $n$ let $[n]$ denote the set of the first $n$ positive integers, $[n] = \{1, 2, \ldots, n\}$. Also let $2^{[n]}$ and $\binom{[n]}{k}$ denote the power set and the collection of all $k$-element subsets of $[n]$, respectively. A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family*, and elements of $\mathcal{F}$ are often called *members*. If $\mathcal{F} \subset \binom{[n]}{k}$, we call it *$k$-uniform*. *Extremal set theory* is a fast developing area within combinatorics which deals with determining or estimating the size $|\mathcal{F}|$ of a family satisfying certain restrictions.

The first result in extremal set theory was Sperner's Theorem, establishing the maximum size of an antichain, i.e., a family without a pair of members one containing the other.

**Theorem 1.1** *(Sperner [138]). Suppose that $\mathcal{A} \subset 2^{[n]}$ satisfies $A \not\subset A'$ for all $A, A' \in \mathcal{A}$. Then it follows $|\mathcal{A}| \le \binom{n}{\lfloor n/2 \rfloor}$. Moreover, the only families achieving equality are $\binom{[n]}{\lfloor n/2 \rfloor}$ and $\binom{[n]}{\lceil n/2 \rceil}$. (Note that for $n$ even they coincide.)*

Sperner's Theorem dates back to 1928 but it remained an isolated result for decades. It was mostly due to the pioneering work of Paul Erdős that systematic research of similar problems started in the 1960s. Erdős' application of Sperner's theorem to Littlewood–Offord problem was also a very early result in Extremal Set Theory. By now extensions and analogues of Sperner's Theorem are very numerous and have been the subject of several survey articles and monographs, cf. e.g., [37]. For this reason we limit the scope of the present survey to another subfield *intersection theorems*. The first instance is the following.

**Theorem 1.2** *(Erdős–Ko–Rado [44]). Let $n > k > t > 0$ be integers and let $\mathcal{F} \subset \binom{[n]}{k}$ satisfy $|F \cap F'| \ge t$ for all $F, F' \in \mathcal{F}$. Then (i) and (ii) hold.*

(i) *If $t = 1$ and $n \ge 2k$ then*

$$|\mathcal{F}| \le \binom{n-1}{k-1}. \tag{1}$$