Journal of Combinatorial Theory, Series A 144 (2016) 212–249

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Lagrange inversion

Ira M. Gessel

Department of Mathematics, Brandeis University, Waltham, MA 02453-2700, United States

A R T I C L E   I N F O

Article history:
Available online 15 July 2016

Keywords:
Lagrange inversion
Compositional inverse
Formal power series
Trees

A B S T R A C T

We give a survey of the Lagrange inversion formula, including different versions and proofs, with applications to combinatorial and formal power series identities.

1. Introduction

The Lagrange inversion formula is one of the fundamental formulas of combinatorics. In its simplest form it gives a formula for the power series coefficients of the solution $f(x)$ of the function equation $f(x) = xG(f(x))$ in terms of coefficients of powers of G. Functional equations of this form often arise in combinatorics, and our interest is in these applications rather than in other areas of mathematics.

There are many generalizations of Lagrange inversion: multivariable forms [25], q-analogues [22,23,28,71], noncommutative versions [6,7,23,56] and others [26,43,45]. In this paper we discuss only ordinary one-variable Lagrange inversion, but in greater detail than elsewhere in the literature.

E-mail address: gessel@brandeis.edu.

http://dx.doi.org/10.1016/j.jcta.2016.06.018

In section 2 we give a thorough discussion of some of the many different forms of Lagrange inversion, prove that they are equivalent to each other, and work through some simple examples involving Catalan and ballot numbers. We address a number of subtle issues that are overlooked in most accounts of Lagrange inversion (and which some readers may want to skip). In section 3 we describe applications of Lagrange inversion to identities involving binomial coefficients, Catalan numbers, and their generalizations. In section 4, we give several proofs of Lagrange inversion, some of which are combinatorial.

A number of exercises giving additional results are included.

An excellent introduction to Lagrange inversion can be found in Chapter 5 of Stanley's *Enumerative Combinatorics*, vol. 2. Other expository accounts of Lagrange inversion can be found in Hofbauer [35], Bergeron, Labelle, and Leroux [5, Chapter 3], Sokal [68], and Merlini, Sprugnoli, and Verri [51].

1.1. Formal power series

Although Lagrange inversion is often presented as a theorem of analysis (see, e.g., Whittaker and Watson [76, pp. 132–133]), we will work only with formal power series and formal Laurent series. A good account of formal power series can be found in Niven [55]; we sketch here some of the basic facts. Given a coefficient ring C, which for us will always be an integral domain containing the rational numbers, the ring $C[[x]]$ of *formal power series* in the variable x with coefficients in C is the set of all "formal sums" $\sum_{n=0}^{\infty} c_n x^n$, where $c_n \in C$, with termwise addition and multiplication defined as one would expect using distributivity: $\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} c_n x^n$, where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$. Differentiation of formal power series is also defined termwise. A series $\sum_{n=0}^{\infty} c_n x^n$ has a multiplicative inverse if and only if c_0 is invertible in C. We may also consider the ring of *formal Laurent series* $C((x))$ whose elements are formal sums $\sum_{n=n_0}^{\infty} c_n x^n$ for some integer n_0, i.e., formal sums $\sum_{n=-\infty}^{\infty} c_n x^n$ in which only finitely many negative powers of x have nonzero coefficients. Henceforth we will omit the word "formal" and speak of power series and Laurent series.

We can iterate the power series and Laurent series ring constructions, obtaining, for example the ring $C((x))[[y]]$ of power series in y whose coefficients are Laurent series in x. In any (possibly iterated) power series or Laurent series ring we will say that a set $\{f_\alpha\}$ of series is *summable* if for any monomial m in the variables, the coefficient of m is nonzero in only finitely many f_α. In this case the sum $\sum_\alpha f$ is well-defined and we will say that $\sum_\alpha f_\alpha$ is summable. If we write $\sum_\alpha f_\alpha$ as an iterated sum, then the order of summation is irrelevant. If $f(x) = \sum_n c_n x^n$ is a Laurent series in $C((x))$ and $u \in C$, where C may be a power series or Laurent series ring, then we say that the substitution of u for x is *admissible* if $f(u) = \sum_n c_n u^n$ is summable, and similarly for multivariable substitutions. Admissible substitutions are homomorphisms. If u is a power series or Laurent series $g(x)$ then $f(g(x))$, if summable, is called the *composition* of f and g. If $f(x) = c_1 x + c_2 x^2 + \cdots$, where c_1 is invertible in C, then there is a unique power series $g(x) = c_1^{-1} x + \cdots$ such that $f(g(x)) = x$; this implies that $g(f(x)) = x$. We call $g(x)$ the