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On the correlation of increasing families

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ABSTRACT

The classical correlation inequality of Harris asserts that any two monotone increasing families on the discrete cube are nonnegatively correlated. In 1996, Talagrand [19] established a lower bound on the correlation in terms of how much the two families depend simultaneously on the same coordinates. Talagrand's method and results inspired a number of important works in combinatorics and probability theory.

In this paper we present stronger correlation lower bounds that hold when the increasing families satisfy natural regularity or symmetry conditions. In addition, we present several new classes of examples for which Talagrand's bound is tight.

A central tool in the paper is a simple lemma asserting that for monotone events noise decreases correlation. This lemma gives also a very simple derivation of the classical FKG inequality

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for product measures, and leads to a simplification of part of Talagrand's proof.

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1. Introduction

Definition 1.1. Let Ω_n denote the discrete cube $\{0,1\}^n$, and identify elements of Ω_n with subsets of $[n] = \{1, 2, \ldots, n\}$ in the natural manner. A family $\mathcal{A} \subset \Omega_n$ is *increasing* if $(S \in \mathcal{A}) \wedge (S \subset T)$ implies $T \in \mathcal{A}$ (alternatively, if the characteristic function $\mathbf{1}_{\mathcal{A}}$ is non-decreasing with respect to the natural partial order on Ω_n).

One of the best-known correlation inequalities is Harris' inequality [10] which asserts that any two increasing families $\mathcal{A}, \mathcal{B} \subset \Omega_n$ are nonnegatively correlated, i.e., satisfy

$$\mathrm{Cov}(\mathcal{A}, \mathcal{B}) = \mu(\mathcal{A} \cap \mathcal{B}) - \mu(\mathcal{A})\mu(\mathcal{B}) \geq 0,$$

where μ is the uniform measure on Ω_n. In 1996, Talagrand [19] presented a lower bound on the correlation, in terms of *influences* of the variables on $\mathcal{A}, \mathcal{B}$.

Definition 1.2. The *influence* of the kth variable on $\mathcal{A} \subset \Omega_n$ is

$$I_k(\mathcal{A}) = 2\mu(\{x \in \mathcal{A} | x \oplus e_k \notin \mathcal{A}\}),$$

where $x \oplus e_k$ is obtained from x by replacing x_k by $1 - x_k$. The *total influence* of $\mathcal{A}$ is $I(\mathcal{A}) = \sum_{k=1}^n I_k(\mathcal{A})$.

We also write $\mathcal{W}_1(\mathcal{A}, \mathcal{B}) = \sum_{i=1}^n I_i(\mathcal{A}) I_i(\mathcal{B})$.

Theorem 1.3 *(Talagrand). Let $\mathcal{A}, \mathcal{B} \subset \Omega_n$ be increasing. Then*

$$\mathrm{Cov}(\mathcal{A}, \mathcal{B}) \geq c \sum_{i=1}^{n} \frac{I_i(\mathcal{A}) I_i(\mathcal{B})}{\log(e / \sum_{i=1}^n I_i(\mathcal{A}) I_i(\mathcal{B}))} = c\varphi\left(\mathcal{W}_1(\mathcal{A}, \mathcal{B})\right), \tag{1}$$

where $\varphi(x) = x / \log(e/x)$ and c is a universal constant.

Talagrand's theorem and the central lemma used in its proof (Lemma 2.7 below) were used in several subsequent works in combinatorics and probability theory (e.g., [1,9,13,20]), most notably in the BKS noise sensitivity theorem [2].

So far, only two classes of tightness examples for Talagrand's lower bound are known. Talagrand [19] showed that his lower bound is tight when $\mathcal{A}, \mathcal{B}$ are increasing Hamming

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