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Smith normal form in combinatorics ☆

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ABSTRACT

This paper surveys some combinatorial aspects of Smith normal form, and more generally, diagonal form. The discussion includes general algebraic properties and interpretations of Smith normal form, critical groups of graphs, and Smith normal form of random integer matrices. We then give some examples of Smith normal form and diagonal form arising from (1) symmetric functions, (2) a result of Carlitz, Roselle, and Scoville, and (3) the Varchenko matrix of a hyperplane arrangement.

1. Introduction

Let A be an $m \times n$ matrix over a field K. By means of elementary row and column operations, namely:

(1) add a multiple of a row (respectively, column) to another row (respectively, column), or
(2) multiply a row or column by a unit (nonzero element) of K,

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we can transform A into a matrix that vanishes off the main diagonal (so A is a diagonal matrix if $m = n$) and whose main diagonal consists of k 1's followed by $m - k$ 0's. Moreover, k is uniquely determined by A since $k = \mathrm{rank}(A)$.

What happens if we replace K by another ring R (which we always assume to be commutative with identity 1)? We allow the same row and column operations as before. Condition (2) above is ambiguous since a unit of R is not the same as a nonzero element. We want the former interpretation, i.e., we can multiply a row or column by a unit only. Equivalently, we transform A into a matrix of the form PAQ, where P is an $m \times m$ matrix and Q is an $n \times n$ matrix, both invertible over R. In other words, $\det P$ and $\det Q$ are units in R. Now the situation becomes much more complicated.

We say that PAQ is a *diagonal form* of A if it vanishes off the main diagonal. (Do not confuse the diagonal form of a square matrix with the matrix D obtained by diagonalizing A. Here $D = XAX^{-1}$ for some invertible matrix X, and the diagonal entries are the eigenvalues of A.) If A has a diagonal form B whose main diagonal is $(\alpha_1, \dots, \alpha_r, 0, \dots, 0)$, where α_i divides α_{i+1} in R for $1 \le i \le r-1$, then we call B a *Smith normal form* (SNF) of A. If A is a nonsingular square matrix, then taking determinants of both sides of the equation $PAQ = B$ shows that $\det A = u\alpha_1 \cdots \alpha_n$ for some unit $u \in R$. Hence an SNF of A yields a factorization of $\det A$. Since there is a huge literature on determinants of combinatorially interesting matrices (e.g., [25,26]), finding an SNF of such matrices could be a fruitful endeavor.

In the next section we review the basic properties of SNF, including questions of existence and uniqueness, and some algebraic aspects. In Section 3 we discuss connections between SNF and the abelian sandpile or chip-firing process on a graph. The distribution of the SNF of a random integer matrix is the topic of Section 4. The remaining sections deal with some examples and open problems related to the SNF of combinatorially defined matrices.

We will state most of our results with no proof or just the hint of a proof. It would take a much longer paper to summarize all the work that has been done on computing SNF for special matrices. We therefore will sample some of this work based on our own interests and research. We will include a number of open problems which we hope will stir up some further interest in this topic.

2. Basic properties

In this section we summarize without proof the basic properties of SNF. We will use the following notation. If A is an $m \times n$ matrix over a ring R, and B is the matrix with $(\alpha_1, \dots, \alpha_m)$ on the main diagonal and 0's elsewhere then we write $A \overset{\mathrm{snf}}{\to} (\alpha_1, \dots, \alpha_m)$ to indicate that B is an SNF of A.

2.1. Existence and uniqueness

For connections with combinatorics we are primarily interested in the ring $\mathbb{Z}$ or in polynomial rings over a field or over $\mathbb{Z}$. However, it is still interesting to ask over what