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Symmetric matrices, Catalan paths, and correlations

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A B S T R A C T

Kenyon and Pemantle (2014) gave a formula for the entries of a square matrix in terms of connected principal and almost-principal minors. Each entry is an explicit Laurent polynomial whose terms are the weights of domino tilings of a half Aztec diamond. They conjectured an analogue of this parametrization for symmetric matrices, where the Laurent monomials are indexed by Catalan paths. In this paper we prove the Kenyon–Pemantle conjecture, and relate this to a statistics problem pioneered by Joe (2006). Correlation matrices are represented by an explicit bijection from the cube to the elliptope.

1. Introduction

In this paper we present a formula for each entry of a symmetric $n \times n$ matrix $X = (x_{ij})$ as a Laurent polynomial in $\binom{n+1}{2}$ distinguished minors of X. Our result verifies a conjecture of Kenyon and Pemantle from [3]. Let I and J be subsets of $[n] = \{1, 2, \ldots, n\}$ with $|I| = |J|$. Let X_I^J denote the minor of X with row indices I and column indices J.

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Here the indices in I and J are always taken in increasing order. The following signed minors will be used:

$$p_I := (-1)^{\lfloor |I|/2 \rfloor} \cdot X_I^I$$
$$\text{and} \quad a_{ij|I} := (-1)^{\lceil |I|/2 \rceil} \cdot X_{iI}^{jI} \quad \text{for} \;\; i, j \notin I, \quad i \neq j.$$

Here $jI := \{j\} \cup I$. We call p_I and $a_{ij|I}$ the *principal* and *almost-principal* minors, respectively. The minors p_I, $a_{ij|I}$ and $a_{ji|I}$ are called *connected* if $1 \leq i < j \leq n$ and $I = \{i+1, i+2, \ldots, j-2, j-1\}$. Note that p_I is not connected when 1 or n is in I. The 1×1-minors $a_{ij} := a_{ij|\emptyset} = x_{ij}$ and $p_k = x_{kk}$ are connected when $|i - j| = 1$ and $1 \leq k \leq n$.

These definitions make sense for every $n \times n$ matrix X, even if X is not symmetric. A general $n \times n$ matrix X has 2^n principal minors, of which $\binom{n-2}{2} + n$ are connected. It also has $n(n-1)2^{n-2}$ almost-principal minors, of which $n(n-1)$ are connected. A symmetric $n \times n$ matrix has $\binom{n}{2}2^{n-2}$ distinct almost-principal minors $a_{ij|I}$, of which $\binom{n}{2}$ are connected.

A *Catalan path* C is a path in the xy-plane which starts at $(0,0)$ and ends on the x-axis, always stays at or above the x-axis, and consists of steps northeast $(1,1)$ and southeast $(1,-1)$. We say that C has *size* n if its endpoints have distance $2n-2$ from each other. Let $\mathscr{C}_n$ denote the set of Catalan paths of size n. Its cardinality equals the Catalan number

$$|\mathscr{C}_n| = \frac{1}{n}\binom{2n-2}{n-1}, \quad \text{which is } 1, 2, 5, 14, 42, 132, 429, 1430, 4862 \quad \text{for } n = 2, \ldots, 10.$$

Let G_n denote the planar graph whose nodes are the $\binom{n+1}{2}$ lattice points (x, y) with $x \geq y \geq 0$ and $x + y \leq 2n-2$ even, and edges are northeast and southeast steps. Thus $\mathscr{C}_n$ consists of the paths from $(0,0)$ to $(2n-2, 0)$ in G_n. We label the nodes and regions of G_n as follows. We assign the label j to the node $(2j-2, 0)$, the label $a_{ij|I}$ to the node $(i+j-2, j-i)$, and the label p_I to the region below that node. Here, $I = \{i+1, i+2, \ldots, j-1\}$. Thus, in the planar graph G_n, the connected principal and almost-principal minors of X are identified with the regions and nodes that are strictly above the x-axis.

The *weight* $W_{\mathscr{C}}(C)$ of a Catalan path C is a Laurent monomial, derived from the drawing of C in the graph G_n. Its numerator is the product of the labels $a_{ij|I}$ of the nodes of G_n that are local maxima or local minima of C, and its denominator is the product of the labels p_I of the regions which are either immediately below a local maximum or immediately above a local minimum. Thus $W_{\mathscr{C}}(C)$ is a Laurent monomial of degree ≤ 1. There is no lower bound on the degree due to minima on the x-axis; for instance, $\frac{a_{13|2}a_{35|4}a_{57|6}a_{79|8}}{p_2p_3p_4p_5p_6p_7p_8}$ has degree -3 and appears for $n = 9$, associated to the path $UUDDUUDDUUDD$.