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Parking functions for mappings

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A B S T R A C T

We apply the concept of parking functions to functional digraphs of mappings by considering the nodes as parking spaces and the directed edges as one-way streets: Each driver has a preferred parking space and starting with this node he follows the edges in the graph until he either finds a free parking space or all reachable parking spaces are occupied. If all drivers are successful we speak of a parking function for the mapping. We transfer well-known characterizations of parking functions to mappings. Via analytic combinatorics techniques we study the total number $M_{n,m}$ of mapping parking functions, i.e., the number of pairs (f, s) with $f : [n] \to [n]$ an n-mapping and $s \in [n]^m$ a parking function for f with m drivers, yielding exact and asymptotic results. Moreover, we describe the phase change behaviour appearing at $m = n/2$ for $M_{n,m}$ and relate it to previously studied combinatorial contexts.

1. Introduction

Parking functions were originally introduced by Konheim and Weiss [14] during their studies of the linear probing collision resolution scheme for hash tables. An illustrative

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definition can be given as follows: Consider a one-way street with n parking spaces numbered from 1 to n and a sequence of m drivers with preferred parking spaces $s_1, s_2, \ldots, s_m$. The drivers arrive sequentially and each driver k, $1 \leq k \leq m$, tries to park at his preferred parking space with address $s_k \in [n]$, where $[n] := \{1, 2, \ldots, n\}$. If it is free he parks. Otherwise he moves further in the allowed direction until he finds a free parking space. If there is no such parking space he leaves the street without parking. A sequence $(s_1, \ldots, s_m) \in [n]^m$ of addresses such that all drivers are able to park is then called a parking function. There are exactly $P_{n,m} = (n+1-m) \cdot (n+1)^{m-1}$ parking functions, for n parking spaces and $0 \leq m \leq n$ drivers [14].

Since their introduction, parking functions have been studied extensively and connections to various other combinatorial objects such as forests, hyperplane arrangements, acyclic functions and non-crossing partitions have been revealed [19]. Moreover, the notion of parking functions has been generalized in several ways, yielding, e.g., (a, b)-parking functions [22], bucket parking functions [2], x-parking functions [20], or G-parking functions [18].

Another natural generalization that has however not been studied yet is the following: Instead of considering simple one-way streets we allow road networks that are modelled by arbitrary directed graphs in which there is always exactly one possibility of moving forward, i.e., graphs in which every node has out-degree 1. Such graphs are the functional digraphs of mappings: Given a mapping $f : [n] \to [n]$ for some positive integer n, its functional digraph $G_f = (V, E)$ is defined on the vertices $V = [n]$ and has the edge set $E = \{(i, f(i)) : i \in [n]\}$. By considering the vertices as parking spaces and the edges one-way streets we obtain a natural generalization of parking functions to mappings. Again, every one of the $0 \leq m \leq n$ drivers has his preferred parking space $s_k \in [n]$ for $k \in [m]$ in the graph. The drivers arrive sequentially and each driver tries to park at his preferred parking space with address s_k. If it is empty he will park, otherwise he follows the edges and parks at the first empty node, if such one exists. Otherwise he cannot park since he would be caught in an endless loop. A pair (f, s) is then called an (n, m)-mapping parking function, if f is an n-mapping and $s \in [n]^m$ is a sequence of addresses such that all m drivers can park in the graph G_f. In Fig. 1 we give an example of a mapping parking function.

To each (n, m)-mapping parking function (f, s) we associate its *output*-function $\pi = \pi_{(f,s)}$, with $\pi : [m] \to [n]$, where $\pi(k)$ is the address of the parking space in which the k-th driver ends up parking. Of course, π is an injection and for the particular case $m = n$ a bijection; thus in the latter case one may speak about the output-permutation π.

Obviously, the concept of mapping parking functions generalizes ordinary parking functions. Every ordinary parking function on $[n]$ can be identified with a parking function for the chain f that maps i to $(i + 1)$ for every $i \in [n - 1]$ and $f(n) = n$. Moreover, as a special case of mappings, we obtain parking functions on Cayley trees, i.e. rooted unordered labelled trees. To see this, let us recall the combinatorial structure of functional digraphs [10]: The weakly connected components are cycles of Cayley trees. That is, each connected component consists of Cayley trees (with edges oriented towards the