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Is there a symmetric version of Hindman's Theorem?

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We show that there does not exist a symmetric version of Hindman's Theorem, or more explicitly, that the property of containing a symmetric IP-set is not divisible. We consider several related dynamics questions.

1. IP and SIP sets

We will use $\mathbb{Z}, \mathbb{Z}_+, \mathbb{N}$ to stand for the sets of integers, nonnegative integers and positive integers, respectively.

For F a finite subset of $\mathbb{Z}$, we denote by $\sigma_F \in \mathbb{Z}$ the sum of the elements of F with the convention that $\sigma_\emptyset = 0$. Of course, if F is a nonempty subset of $\mathbb{N}$, then $\sigma_F \in \mathbb{N}$.

Call a subset A of $\mathbb{Z}$ *symmetric* if $-A = A$ where $-A = \{-a : a \in A\}$. For any subset A of $\mathbb{Z}$ let $A_\pm = A \cup -A$ so that $A_\pm$ is the smallest symmetric set which contains A. On the other hand, let $A_+ = A \cap \mathbb{N}$, the positive part of A. Note that if A is symmetric then $A = A_\pm$ and $A \setminus \{0\} = (A_+)_\pm$.

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For subsets A_1, A_2 of $\mathbb{Z}$ we let $A_1 + A_2 = \{a_1 + a_2 : a_1 \in A_1, a_2 \in A_2\}$ and $A_1 - A_2 = A_1 + (-A_2)$. If $A_2 = \{n\}$ we write $A_1 - n$ for $A_1 - A_2$.

Let A be a nonempty subset of $\mathbb{Z}$. We set

$$D(A) = \{a_1 - a_2 : a_1, a_2 \in A\} = A - A$$

$$\mathrm{IP}(A) = \{\sigma_F : F \text{ a finite subset of } A\}$$

$$\mathrm{SIP}(A) = D(\mathrm{IP}(A)) = \mathrm{IP}(A) - \mathrm{IP}(A).$$

Clearly, $0 \in D(A)$ and $D(A)$ is symmetric and so the same is true of $\mathrm{SIP}(A)$. If $0 \in A$ then $A \subset D(A)$. In general, $D(A) \cup A \cup -A = D(A \cup \{0\})$. In particular, $0 = \sigma_\emptyset \in \mathrm{IP}(A)$ implies $\mathrm{IP}(A) \subset \mathrm{SIP}(A)$.

If $A \subset \mathbb{N}$ then $\mathrm{IP}(A) = \{0\} \cup \mathrm{IP}(A)_+$ since $0 = \sigma_\emptyset \in \mathrm{IP}(A)$. $\mathrm{IP}(A) = \mathrm{IP}(A \cup \{0\}) = \mathrm{IP}(A \setminus \{0\})$.

N.B. We include $0 = \sigma_\emptyset$ in $\mathrm{IP}(A)$ for A any nonempty subset of $\mathbb{Z}$. This is a convenience which ensures, for example, that $\mathrm{IP}(A)$ is a subset of $\mathrm{SIP}(A)$ for $A \subset \mathbb{N}$. For $A \subset \mathbb{N}$, our set $\mathrm{IP}(A)_+$ is what is more often called the IP set on A, following Definition 2.3 of [2].

Lemma 1.1. *If B is a nonempty subset of $\mathbb{N}$ then*

$$\mathrm{SIP}(B) = \mathrm{IP}(B_\pm).$$

If A is a symmetric subset of $\mathbb{Z}$ with $A \setminus \{0\}$ nonempty then

$$\mathrm{SIP}(A_+) = \mathrm{IP}(A).$$

Proof. If $F \subset B_\pm$ then

$$\sigma_F = \sigma_{F\cap B} + \sigma_{F\cap -B} = \sigma_{F\cap B} - \sigma_{(-F)\cap B}.$$

Hence, $\mathrm{IP}(B_\pm) \subset \mathrm{SIP}(B)$.

For the reverse inclusion, let F_1, F_2 be finite subsets of B.

$$\sigma_{F_1} - \sigma_{F_2} = \sigma_{F_1 \cup -F_2},$$

since F_1 and $-F_2$ are disjoint.

If A is symmetric and $A \setminus \{0\}$ is nonempty then A_+ is nonempty and the previous result applied to $B = A_+$ yields the second equation since $(A_+)_\pm = A \setminus \{0\}$ and $\mathrm{IP}(A) = \mathrm{IP}(A \setminus \{0\})$. □

We say that a subset $B \subset \mathbb{N}$ is

- a *difference set* if there exists an infinite subset A of $\mathbb{N}$ such that $D(A)_+ \subset B$.
- an IP *set* if there exists an infinite subset A of $\mathbb{N}$ such that $\mathrm{IP}(A)_+ \subset B$.
- an SIP *set* if there exists an infinite subset A of $\mathbb{N}$ such that $\mathrm{SIP}(A)_+ \subset B$.