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Counting permutations by runs

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ABSTRACT

In his Ph.D. thesis, Ira Gessel proved a reciprocity formula for noncommutative symmetric functions which enables one to count words and permutations with restrictions on the lengths of their increasing runs. We generalize Gessel's theorem to allow for a much wider variety of restrictions on increasing run lengths, and use it to complete the enumeration of permutations with parity restrictions on peaks and valleys, and to give a systematic method for obtaining generating functions for permutation statistics that are expressible in terms of increasing runs. Our methods can also be used to obtain analogous results for alternating runs in permutations.

1. Introduction

Given a set A, let A^* be the set of all finite sequences of elements of A, including the empty sequence. We call A an *alphabet*, the elements of A *letters*, A^* the *free monoid* on A,[1] and the elements of A^* *words*. If we refer to words without specifying an alphabet, then we take the alphabet to be $\mathbb{P}$, the set of positive integers. Suppose that our alphabet A is a totally ordered set, such as $\mathbb{P}$. Then every word in A^* can be uniquely decomposed

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1 Indeed, A^* is a free monoid generated by the letters in A, under the operation of concatenation.

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into a sequence of maximal weakly increasing consecutive subsequences, which we call *increasing runs*. For example, the increasing runs of 2142353 are 2, 14, 235, and 3. The notion of increasing runs clearly extends to permutations as well.

As part of his 1977 Ph.D. thesis, Gessel proved a very general identity involving noncommutative symmetric functions [6, Theorem 5.2] from which we can obtain many generating functions for words and permutations with restrictions on the lengths of their increasing runs. We call this result the "run theorem". In this paper, we present a generalization of Gessel's run theorem that will enable us to count words and permutations with an even wider variety of restrictions on increasing run lengths. Specifically, these restrictions are those which can be encoded by a special type of digraph that we shall call a "run network".

The organization of this paper is as follows. In Section 2, we introduce some preliminary definitions, state Gessel's run theorem, and present our generalization of the run theorem. In Sections 3 and 4, we present two separate applications of the generalized run theorem to permutation enumeration.

Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation in $\mathfrak{S}_n$, the set of permutations of $[n] = \{1, 2, \ldots, n\}$ (or more generally, any sequence of n distinct integers); such permutations are called *n-permutations*. We say that i is a *peak* of π if $\pi_{i-1} < \pi_i > \pi_{i+1}$ and that i is a *valley* of π if $\pi_{i-1} > \pi_i < \pi_{i+1}$. For example, given $\pi = 5736214$, its peaks are 2 and 4, and its valleys are 3 and 6.

In [9], Gessel and Zhuang found the exponential generating function

$$\frac{3\sin\left(\frac{1}{2}x\right) + 3\cosh\left(\frac{1}{2}\sqrt{3}x\right)}{3\cos\left(\frac{1}{2}x\right) - \sqrt{3}\sinh\left(\frac{1}{2}\sqrt{3}x\right)} \tag{1.1}$$

for permutations with all peaks odd and all valleys even. Amazingly, (1.1) can also be expressed as

$$\left(1 - E_1x + E_3\frac{x^3}{3!} - E_4\frac{x^4}{4!} + E_6\frac{x^6}{6!} - E_7\frac{x^7}{7!} + \cdots\right)^{-1} \tag{1.2}$$

where the Euler numbers E_n are defined by the identity $\sum_{n=0}^{\infty} E_n x^n/n! = \sec x + \tan x$, which is reminiscent of David and Barton's [3] generating function

$$\left(1 - x + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \cdots\right)^{-1} \tag{1.3}$$

for permutations with no increasing runs of length 3 or greater. The authors explained the similarity between these two generating functions by applying two different homomorphisms to an identity obtained by the run theorem, which we review in Section 2.2 after introducing the run theorem.

The generating function (1.2) also counts permutations with all peaks even and all valleys odd, because these permutations are in bijection with permutations with all peaks