Journal of Combinatorial Theory, Series A 142 (2016) 177–180

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Polytopes with few vertices and few facets ☆

Arnau Padrol

Institut de Mathématiques de Jussieu-Paris Rive Gauche (UMR 7586), Sorbonne Universités, Université Pierre et Marie Curie (Paris 6), 4 place Jussieu, 75252 Paris Cedex 5, France

ARTICLE INFO

Article history:
Received 12 April 2015
Available online 4 May 2016

Keywords:
Convex polytopes
Number of combinatorial types
Perles' Skeleton Theorem

ABSTRACT

In this note we prove that the number of combinatorial types of d-polytopes with $d + 1 + \alpha$ vertices and $d + 1 + \beta$ facets is bounded by a constant independent of d.

There is only one combinatorial type of d-polytope with $d + 1$ vertices, the simplex. Every d-polytope with $d + 2$ vertices is combinatorially equivalent to a repeated pyramid over a direct sum of two simplices, $\mathrm{pyr}_k(\Delta_n \oplus \Delta_m)$ with $k \geq 0$, $m, n \geq 1$ (cf. [2, Section 6.1]). The number of d-polytopes with $d + 3$ vertices is exponential in d [1], and the number of d-polytopes with $d + 4$ vertices is already super-exponential in d [8,9]. Of course, the same numbers apply for polytopes with few facets, by polarity.

In this note we prove that, in contrast, there are few (combinatorial types of) convex polytopes that have both few vertices and few facets.

☆ This research was supported by the DFG Collaborative Research Center SFB/TR 109 "Discretization in Geometry and Dynamics".

E-mail address: arnau.padrol@imj-prg.fr.

http://dx.doi.org/10.1016/j.jcta.2016.04.001
0097-3165/

Theorem 1. *For each pair of nonnegative integers α and β there is a constant $K(\alpha, \beta)$, independent from d, such that the number of combinatorial types of d-polytopes with no more than $d+1+\alpha$ vertices and no more than $d+1+\beta$ facets is bounded above by $K(\alpha, \beta)$.*

This theorem is a direct consequence of the following structural result.

Theorem 2. *For each pair of nonnegative integers α and β there is a constant $D(\alpha, \beta) = \mathcal{O}(\alpha^2 + \beta^2)$ such that every d-polytope with no more than $d+1+\alpha$ vertices and no more than $d+1+\beta$ facets is a join of a simplex and an at most $D(\alpha, \beta)$-dimensional polytope.*

Equivalently, every d-polytope with $d > D(\alpha, \beta)$ either is a pyramid, has more than $d+1+\alpha$ vertices or has more than $d+1+\beta$ facets.

Moreover, (the minimal such) $D(\alpha, \beta)$ satisfies

$$\varphi(\alpha, \beta) \leq D(\alpha, \beta) \leq \min\{\Phi(\alpha, \beta), \Phi(\beta, \alpha)\},$$

where

$$\Phi(x, y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0, \\ 3x + y - 2 & \text{if } 1 \leq x \leq 5, \\ \binom{x}{2} + y + 3 & \text{if } x \geq 5; \end{cases}$$

and

$$\varphi(x, y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0, \\ x + y & \text{if } x = 1 \text{ or } y = 1, \\ x + 2y - 1 & \text{if } x \geq y > 1, \\ 2x + y - 1 & \text{if } y \geq x > 1. \end{cases}$$

Indeed, this theorem shows that for every d the number of combinatorial types of d-polytopes with no more than $d+1+\alpha$ vertices and no more than $d+1+\beta$ facets is bounded above by those in dimension $D(\alpha, \beta)$. Since the vertex-facet incidences determine the combinatorial type, we get the following crude estimate for $K(\alpha, \beta)$:

$$K(\alpha, \beta) < 2^{(D(\alpha,\beta)+\alpha+1)(D(\alpha,\beta)+\beta+1)} = 2^{\mathcal{O}(\alpha^4+\beta^4)}.$$

Our proof of Theorem 2 is based on a result of Marcus on minimal positively 2-spanning configurations [6,7], which via Gale duality provides lower bounds on the number of vertices of what Wotzlaw and Ziegler call *unneighborly polytopes* [11]. A polytope P is *unneighborly* if for every vertex v of P there is another vertex w such that (v, w) does not form an edge of the graph of P.

Theorem 3 *(Marcus 1981 [6]). If P is an unneighborly d-polytope with $d+1+\alpha$ vertices, then*

$$d \leq \begin{cases} 3\alpha - 1 & \text{if } \alpha \leq 5, \\ \binom{\alpha}{2} + 4 & \text{if } \alpha \geq 5. \end{cases}$$