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Some spectral and quasi-spectral characterizations of distance-regular graphs

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ABSTRACT

In this paper we consider the concept of preintersection numbers of a graph. These numbers are determined by the spectrum of the adjacency matrix of the graph, and generalize the intersection numbers of a distance-regular graph. By using the preintersection numbers we give some new spectral and quasi-spectral characterizations of distance-regularity, in particular for graphs with large girth or large odd-girth.

1. Introduction

A central issue in spectral graph theory is to question whether or not a graph is uniquely determined by its spectrum, see the surveys of Van Dam and Haemers [12,13]. In particular, much attention has been paid to give spectral and quasi-spectral characterizations of distance-regularity. Contributions in this area are due to Brouwer and

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Haemers [2], Van Dam and Haemers [11], Van Dam, Haemers, Koolen, and Spence [15], Haemers [21], and Huang and Liu [23], among others.

In this paper, we will give new spectral and quasi-spectral characterizations of distance-regularity of a graph G without requiring, as it is common in this area of research, that:

- G is cospectral with a distance-regular graph Γ, where
- Γ has intersection numbers, or other combinatorial parameters that satisfy certain properties.

The following theorem, given in the recent survey by Van Dam, Koolen, and Tanaka [16] (see also Van Dam and Haemers [12] and Brouwer and Haemers [3]), contains most of the known characterizations of this type.

Theorem 1.1. *If Γ is a distance-regular graph with diameter $D = d$ and girth g satisfying one of the properties (i)–(v), then every graph G cospectral with Γ is also distance-regular and has the same intersection numbers as Γ.*

(i) $g \geq 2d - 1$,
(ii) $g \geq 2d - 2$ *and* Γ *is bipartite,*
(iii) $g \geq 2d - 2$ *and* $c_{d-1}c_d < -(c_{d-1} + 1)(\lambda_1 + \cdots + \lambda_d)$,
(iv) Γ *is a generalized Odd graph, i.e.,* $a_1 = \cdots = a_{d-1} = 0$, $a_d \neq 0$,
(v) $c_1 = \cdots = c_{d-1} = 1$.

Here we show that the same conclusions can be obtained within the following much more general setting:

- G has preintersection numbers satisfying certain properties.

More precisely, in Theorem 4.4 we generalize the cases (i) and (ii) of Theorem 1.1. A refinement of (iii) is given in Theorem 4.12. Moreover, our Theorem 4.1 provides an alternative formulation of the so-called odd-girth theorem, which is a generalization of (iv). Finally, Theorem 4.9 generalizes the case (v).

Our work is motivated by earlier work in this direction, in particular by the odd-girth theorem [14]. This result states that a graph with $d+1$ distinct eigenvalues and odd-girth $2d + 1$ is distance-regular. We recall that the odd-girth of a graph is the length of the shortest odd cycle in the graph, and that the odd-girth follows from the spectrum of the graph.

In order to obtain our results we mainly use the theory on so-called almost distance-regular graphs given by Dalfó, Van Dam, Fiol, Garriga, and Gorissen [7]. Another important ingredient of our work is a new inequality (Proposition 4.6) for partially distance-regular graphs that is inspired by Fiol and Garriga's spectral excess theorem