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Cayley digraphs of 2-genetic groups of odd prime-power order

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ABSTRACT

A group is called 2-*genetic* if each normal subgroup of the group can be generated by two elements. Let G be a non-abelian 2-genetic group of order p^n for an odd prime p and a positive integer n. In this paper, we investigate connected Cayley digraphs $\mathrm{Cay}(G, S)$ for non-abelian 2-genetic groups G of odd order p^n, and determine their full automorphism groups $A = \mathrm{Aut}(\mathrm{Cay}(G, S))$ in the case when $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a p'-group. It is shown that either $\mathrm{Cay}(G, S)$ is normal, that is, the right regular representation of G is normal in A, or $p = 3, 5, 7, 11$ and the largest normal p-subgroup $\mathrm{O}_p(A)$ of A has order p^{n-1} with $\mathrm{ASL}(2, p) \le A/\Phi(\mathrm{O}_p(A)) \le \mathrm{AGL}(2, p)$. Furthermore, a non-normal Cayley digraph with smallest order and smallest valency is constructed for each $p = 3, 5, 7, 11$, respectively. In particular, the underlying graphs of the above non-normal Cayley digraphs for $p = 3, 7, 11$ are half-arc-transitive, and they are the first constructions of half-arc-transitive non-normal Cayley graphs of order a prime-power.

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1. Introduction

For a finite simple digraph (directed graph) Γ, we use $V(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its vertex set and full automorphism group, respectively. For $u, v \in V(\Gamma)$, (u, v) is the arc (directed edge) starting from u and ending at v in Γ, and $A(\Gamma)$ is the arc set of Γ. For a finite simple graph (undirected graph) X, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ the vertex set, the edge set and the full automorphism group of X, respectively. An ordered pair of adjacent vertices in X is called an *arc*, and we denote by $A(X)$, the arc set of X. The graph X is said to be *vertex-transitive*, *edge-transitive* or *arc-transitive* (*symmetric*) if $\mathrm{Aut}(X)$ acts transitively on $V(X)$, $E(X)$ or $A(X)$, respectively, and *half-arc-transitive* if it is vertex-transitive, edge-transitive, but not arc-transitive. Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular.

Let G be a finite group and S a subset of G such that $1 \notin S$. The *Cayley digraph* $\Gamma = \mathrm{Cay}(G, S)$ of G with respect to S is defined as the digraph with vertex set $V(\Gamma) = G$ and arc set $A(\Gamma) = \{(g, sg) \mid g \in G, s \in S\}$. A Cayley digraph $\mathrm{Cay}(G, S)$ is connected if and only if $G = \langle S \rangle$, and if S is symmetric, that is, $S^{-1} = \{s^{-1} \mid s \in S\} = S$, then $\mathrm{Cay}(G, S)$ can be viewed as a graph by identifying the two opposite arcs (g, sg) and (sg, g) as an edge $\{g, sg\}$. Thus, Cayley graph is a special case of Cayley digraph. It is easy to see that $\mathrm{Aut}(\mathrm{Cay}(G, S))$ contains the right regular representation $\hat{G} = \{\hat{g} \mid g \in G\}$ of G, where $\hat{g}$ is the map on G defined by $x \mapsto xg$, $x \in G$, and that $\hat{G}$ is regular on the vertex set $V(\Gamma)$. This implies that a Cayley digraph is vertex-transitive. Also, it is easy to check that $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))_1$, the stabilizer of the vertex 1 in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. A Cayley digraph $\mathrm{Cay}(G, S)$ is said to be *normal* if $\hat{G}$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$.

Determining the automorphism group of a Cayley digraph is fundamental in algebraic graph theory, but very difficult in general. Since a Cayley digraph $\mathrm{Cay}(G, S)$ is defined by G, a natural approach to determine its automorphism group is to understand the relationship between $\mathrm{Aut}(\mathrm{Cay}(G, S))$ and $\hat{G}$, for example, whether or not $\mathrm{Cay}(G, S)$ is normal. In particular, if $\mathrm{Cay}(G, S)$ is normal then $\mathrm{Aut}(\mathrm{Cay}(G, S))$ is a semidirect product of $\hat{G}$ by the subgroup $\mathrm{Aut}(G, S)$ (see Proposition 2.1), and hence $\mathrm{Aut}(\mathrm{Cay}(G, S))$ is completely determined by $\mathrm{Aut}(G)$, which is much easier to determine. Thus a natural problem is to determine normality of Cayley digraphs for a given class of groups.

Wang et al. [29] obtained all disconnected normal Cayley graphs. The normality of Cayley graphs of cyclic groups of order a prime and of groups of order twice a prime was solved by Alspach [1] and Du et al. [12], respectively. Dobson [9] determined all non-normal Cayley graphs of order a product of two distinct primes. For a prime p, Dobson and Witte [11] determined all non-normal Cayley graphs of order p^2, and Dobson and Kovács [10] determined the full automorphism groups of Cayley digraphs of $\mathbb{Z}_p^3$. The normality of Cayley graphs of finite simple groups was investigated by Fang, Praeger and Wang [15], and the normality of Cayley digraphs of finite simple groups of small