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Forbidding Hamilton cycles in uniform hypergraphs ☆

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For $1 \leq d \leq \ell < k$, we give a new lower bound for the minimum d-degree threshold that guarantees a Hamilton ℓ-cycle in k-uniform hypergraphs. When $k \geq 4$ and $d < \ell = k-1$, this bound is larger than the conjectured minimum d-degree threshold for perfect matchings and thus disproves a well-known conjecture of Rödl and Ruciński. Our (simple) construction generalizes a construction of Katona and Kierstead and the space barrier for Hamilton cycles.

1. Introduction

The study of Hamilton cycles is an important topic in graph theory. A classical result of Dirac [4] states that every graph on $n \geq 3$ vertices with minimum degree $n/2$ contains a Hamilton cycle. In recent years, researchers have worked on extending this theorem to hypergraphs – see recent surveys [15,18,26].

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To define Hamilton cycles in hypergraphs, we need the following definitions. Given $k \geq 2$, a k-uniform hypergraph (in short, *k-graph*) consists of a vertex set V and an edge set $E \subseteq \binom{V}{k}$, where every edge is a k-element subset of V. Given a k-graph H with a set S of d vertices (where $1 \leq d \leq k-1$) we define $\deg_H(S)$ to be the number of edges containing S (the subscript H is omitted if it is clear from the context). The *minimum d-degree* $\delta_d(H)$ of H is the minimum of $\deg_H(S)$ over all d-vertex sets S in H. For $1 \leq \ell \leq k-1$, a k-graph is a called an *ℓ-cycle* if its vertices can be ordered cyclically such that each of its edges consists of k consecutive vertices and every two consecutive edges (in the natural order of the edges) share exactly ℓ vertices. In k-graphs, a $(k-1)$-cycle is often called a *tight* cycle. We say that a k-graph contains a *Hamilton ℓ-cycle* if it contains an ℓ-cycle as a spanning subhypergraph. Note that a Hamilton ℓ-cycle of a k-graph on n vertices contains exactly $n/(k-\ell)$ edges, implying that $k-\ell$ divides n.

Let $1 \leq d, \ell \leq k-1$. For $n \in (k-\ell)\mathbb{N}$, we define $h_d^\ell(k,n)$ to be the smallest integer h such that every n-vertex k-graph H satisfying $\delta_d(H) \geq h$ contains a Hamilton ℓ-cycle. Note that whenever we write $h_d^\ell(k,n)$, we always assume that $1 \leq d \leq k-1$. Moreover, we often write $h_d(k,n)$ instead of $h_d^{k-1}(k,n)$ for simplicity. Similarly, for $n \in k\mathbb{N}$, we define $m_d(k,n)$ to be the smallest integer m such that every n-vertex k-graph H satisfying $\delta_d(H) \geq m$ contains a perfect matching. The problem of determining $m_d(k,n)$ has attracted much attention recently and the asymptotic value of $m_d(k,n)$ is conjectured as follows. Note that the $o(1)$ term refers to a function that tends to 0 as $n \to \infty$ throughout the paper.

Conjecture 1.1. [6,14] *For $1 \leq d \leq k-1$ and $k \mid n$,*

$$m_d(k,n) = \left(\max\left\{\frac{1}{2}, 1 - \left(1 - \frac{1}{k}\right)^{k-d}\right\} + o(1)\right)\binom{n-d}{k-d}.$$

Conjecture 1.1 has been confirmed [1,17] for $\min\{k-4, k/2\} \leq d \leq k-1$ (the exact values of $m_d(k,n)$ are also known in some cases, e.g., [23,25]). On the other hand, $h_d^\ell(k,n)$ has also been extensively studied [2,3,5,7–13,16,19,20,22,24]. In particular, Rödl, Ruciński and Szemerédi [20,22] showed that $h_{k-1}(k,n) = (1/2 + o(1))n$. The same authors proved in [21] that $m_{k-1}(k,n) = (1/2 + o(1))n$ (later they determined $m_{k-1}(k,n)$ exactly [23]). This suggests that the values of $h_d(k,n)$ and $m_d(k,n)$ are closely related and inspires Rödl and Ruciński to make the following conjecture.

Conjecture 1.2. [18, Conjecture 2.18] *Let $k \geq 3$ and $1 \leq d \leq k-2$. Then*

$$h_d(k,n) = m_d(k,n) + o(n^{k-d}).$$

By using the value of $m_d(k,n)$ from Conjecture 1.1, Kühn and Osthus stated this conjecture explicitly for the case $d = 1$.