J. Combin. Theory Ser. A 140 (2016) 1–37

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Inequivalent factorizations of permutations

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A R T I C L E   I N F O

Article history:
Received 20 May 2014
Available online 29 December 2015

Keywords:
Symmetric group
Enumeration
Permutation factorization
Planar maps

A B S T R A C T

Two factorizations of a permutation into products of cycles are equivalent if one can be obtained from the other by repeatedly interchanging adjacent disjoint factors. This paper studies the enumeration of equivalence classes under this relation.
We establish general connections between inequivalent factorizations and other well-studied classes of permutation factorizations, such as monotone factorizations. We also obtain several specific enumerative results, including closed form generating series for inequivalent minimal transitive factorizations of permutations having up to three cycles. Our derivations rely on a new correspondence between inequivalent factorizations and acyclic alternating digraphs. Strong similarities between the enumerative results derived here and analogous ones for "ordinary" factorizations suggest that a unified theory remains to be discovered.

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http://dx.doi.org/10.1016/j.jcta.2015.12.002

1. Introduction

1.1. Notation

We adhere to standard notation and terminology concerning permutations. We write $\mathfrak{S}_n$ for the symmetric group on the symbols $\{1, 2, \dots, n\}$, and we multiply permutations from right to left. The number of cycles in $\pi \in \mathfrak{S}_n$ is denoted by $\ell(\pi)$. For a composition $\alpha = (\alpha_1, \dots, \alpha_m)$ of n, we write $\mathcal{C}_\alpha$ for the *conjugacy class* of $\mathfrak{S}_n$ consisting of all permutations whose disjoint cycles are of lengths $\alpha_1, \dots, \alpha_m$. Elements of $\mathcal{C}_\alpha$ are said to be of *cycle type* α. Permutations of cycle type $(k, 1, 1, \dots, 1)$ are called *k-cycles*, with 2-cycles more commonly referred to as *transpositions*. We typically suppress cycles of length 1 when writing permutations in disjoint cycle notation. Thus $(i\,j)$ denotes a transposition in $\mathfrak{S}_n$, with the value of n to be understood from context.

For any list of integers $\beta = (\beta_1, \beta_2, \dots)$ with finite support, let $|\beta| = \sum_k \beta_k$ and let $\ell(\beta)$ be the number of nonzero entries of β. In particular, for $\pi \in \mathcal{C}_\alpha \subset \mathfrak{S}_n$ we have $|\alpha| = n$ and $\ell(\alpha) = \ell(\pi)$.

For an integer partition l and a set of indeterminates $\mathbf{x} = (x_1, \dots, x_n)$, we write $h_l(\mathbf{x})$, $e_l(\mathbf{x})$ and $s_l(\mathbf{x})$, respectively, for the complete, elementary, and Schur symmetric polynomials indexed by l. We adopt the convention that each of these polynomials is 0 when l is not a partition.

The ring of formal power series in indeterminates $\mathbf{x} = (x_1, \dots, x_m)$ over the ring R is denoted by $R[[\mathbf{x}]]$. If $f \in R[[\mathbf{x}]]$ and $\mathbf{i} = (i_1, \dots, i_m)$ is a list of nonnegative integers, then we write $[\mathbf{x}^\mathbf{i}]\, f$ for the coefficient of the monomial $\mathbf{x}^\mathbf{i} = x_1^{i_1} \cdots x_m^{i_m}$ in f. We let $D_\mathbf{x}$ denote the total derivative operator on $R[[\mathbf{x}]]$, namely $D_\mathbf{x} = \sum_{i=1}^m x_i \frac{\partial}{\partial x_i}$.

1.2. Factorizations of permutations

A *factorization* of a permutation $\pi \in \mathfrak{S}_n$ is a tuple $f = (\sigma_1, \dots, \sigma_r)$ where each $\sigma_i \in \mathfrak{S}_n$ and $\pi = \sigma_1 \cdots \sigma_r$. The σ_i are the *factors* of f. The number of factors, r, is the *length* of f, and is denoted by $\ell(f)$. We will generally be less formal and write a factorization simply as the product of its factors. For instance,

$$(1\,2\,3)(4\,6) \cdot (2\,4\,6\,5) \cdot (1\,4)(2\,3)(5\,6) \tag{1}$$

is a factorization of $(1\,4\,2)(3\,6)(5)$ of length 3.

Let f be a factorization of $\pi \in \mathfrak{S}_n$. We define the *class* of f to be the cycle type of π, while the *signature* of f is the list $\beta = (\beta_2, \beta_3, \dots)$, where β_k is the total number of k-cycles amongst all factors. The *depth* of f, denoted by $\langle f \rangle$, is defined as

$$\langle f \rangle = \sum_{j \geq 2} (j-1)\beta_j.$$