J. Combin. Theory Ser. A 134 (2015) 1–35

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

On the topology of the permutation pattern poset

Peter R.W. McNamara [a,1], Einar Steingrímsson [b,2]

[a] *Department of Mathematics, Bucknell University, Lewisburg, PA 17837, USA*
[b] *Department of Computer and Information Sciences, University of Strathclyde, Glasgow G1 1XH, UK*

ARTICLE INFO

Article history:
Received 5 June 2013
Available online 20 March 2015

Keywords:
Pattern poset
Shellable
Disconnected
Layered permutations
Generalized subword order
Möbius function

ABSTRACT

The set of all permutations, ordered by pattern containment, forms a poset. This paper presents the first explicit major results on the topology of intervals in this poset. We show that almost all (open) intervals in this poset have a disconnected subinterval and are thus not shellable. Nevertheless, there seem to be large classes of intervals that are shellable and thus have the homotopy type of a wedge of spheres. We prove this to be the case for all intervals of layered permutations that have no disconnected subintervals of rank 3 or more. We also characterize in a simple way those intervals of layered permutations that are disconnected. These results carry over to the poset of generalized subword order when the ordering on the underlying alphabet is a rooted forest. We conjecture that the same applies to intervals of separable permutations, that is, that such an interval is shellable if and only if it has no disconnected subinterval of rank 3 or more. We also present a simplified version of the recursive formula for the Möbius function of decomposable permutations given by Burstein et al. [9].

E-mail addresses: peter.mcnamara@bucknell.edu (P.R.W. McNamara), einar@alum.mit.edu (E. Steingrímsson).
URLs: http://www.facstaff.bucknell.edu/pm040 (P.R.W. McNamara), http://https://personal.cis.strath.ac.uk/einar.steingrimsson (E. Steingrímsson).

[1] This work was partially supported by a grant from the Simons Foundation (#245597 to Peter McNamara).
[2] Steingrímsson was supported by grant No. 090038013 from the Icelandic Research Fund.

http://dx.doi.org/10.1016/j.jcta.2015.02.009
0097-3165/

1. Introduction

An occurrence of a pattern p in a permutation π is a subsequence of π whose letters appear in the same relative order of size as those in p. For example, the permutation 416325 contains two occurrences of the pattern 231, in 463 and 462. The origin of the study of permutation patterns can be traced back a long way. In the 1960s and 1970s the number of permutations of length n avoiding (having no occurrence of) any one of the six patterns of length 3 was determined by Knuth [16, Exercise 2.2.1.5] and Rogers [20]. In all of these cases, which are easily seen to fall into two equivalence classes, the numbers in question turn out to be the n-th Catalan number. In a seminal 1985 paper, Simion and Schmidt [24] then did the first systematic study of pattern avoidance, and established, among other things, the number of permutations avoiding any given set of patterns of length 3. In the last two decades this research area has grown steadily, and explosively in recent years, with several different directions emerging. Also, many connections to other branches of combinatorics, other mathematics, physics and biology have been developed, in addition to the strong ties to theoretical computer science, in which pattern research also has roots. One of the early such connections was established in 1990 by Lakshmibai and Sandhya [17], who showed that a Schubert variety X_π is smooth if and only if π avoids both 4231 and 3412. For a recent comprehensive survey of pattern research see [15], and [29] for an overview of the latest developments.

It is easy to see that pattern containment defines a poset (partially ordered set) $\mathcal{P}$ on the set of all permutations of length n for all $n > 0$. This poset is the underlying object of all studies of pattern avoidance and containment. A classical question about any combinatorially defined poset is what its Möbius function is, and in [34], Wilf asked what can be said about the Möbius function of $\mathcal{P}$. A generalization of that question concerns the topology of the (order complexes of) intervals in $\mathcal{P}$, since the Möbius function of an interval $I = [a, b]$ in $\mathcal{P}$ equals the reduced Euler characteristic of the topological space determined by the order complex $\Delta(I)$, whose faces are the chains of the open interval (a, b). In particular, we would like to know the homology and the homotopy type of intervals in $\mathcal{P}$.

The first results on the Möbius function of intervals of $\mathcal{P}$ were obtained by Sagan and Vatter [22], who used discrete Morse theory to compute the Möbius function for the poset of layered permutations; as they pointed out, this poset is easily seen to be isomorphic to a certain poset they studied of compositions of an integer. Later results about the Möbius function of $\mathcal{P}$ have been obtained by Steingrímsson and Tenner [30] and by Burstein et al. [9], the latter of which gave an effective formula for the Möbius function of intervals of separable permutations (those avoiding both of the patterns 2413 and 3142) and reduced the computation for decomposable permutations (those non-trivially expressible as direct sums) to that for indecomposable ones. Recently, Smith [25] (see also [26]) obtained the first systematic results for several classes of intervals of indecomposable permutations, including those intervals $[1, \pi]$ where π is any permutation with exactly one descent.