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Large sets of complex and real equiangular lines

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ABSTRACT

Large sets of equiangular lines are constructed from sets of mutually unbiased bases, over both the complex and the real numbers.

1. Introduction

The *angle* between vectors $\boldsymbol{x}_j$ and $\boldsymbol{x}_k$ of unit norm in $\mathbb{C}^d$ is $\arccos |\langle \boldsymbol{x}_j, \boldsymbol{x}_k \rangle|$, where $\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product. A set of m distinct lines in $\mathbb{C}^d$ through the origin, represented by vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ of equal norm, is *equiangular* if for some real constant a we have

$$|\langle \boldsymbol{x}_j, \boldsymbol{x}_k \rangle| = a \quad \text{for all } j \neq k.$$

The number of equiangular lines in $\mathbb{C}^d$ is at most d^2 [4], and when the vectors are further constrained to lie in $\mathbb{R}^d$ this number is at most $d(d+1)/2$ (attributed to Gerzon in [9]).

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It is an open question, in both the complex and real case, whether the upper bound can be attained for infinitely many d, although in both cases $\Theta(d^2)$ equiangular lines exist for all d. Specifically, König [8] constructed $d^2 - d + 1$ equiangular lines in $\mathbb{C}^d$ where $d-1$ is a prime power, and de Caen [3] constructed $2(d+1)^2/9$ equiangular lines in $\mathbb{R}^d$ where $(d+1)/3$ is twice a power of 4. By extending vectors using zero entries as necessary, we can derive sets of $\Theta(d^2)$ equiangular lines from these direct constructions for all d.

Two orthogonal bases $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_d\}$, $\{\boldsymbol{y}_1, \dots, \boldsymbol{y}_d\}$ for $\mathbb{C}^d$ are *unbiased* if

$$\frac{|\langle \boldsymbol{x}_j, \boldsymbol{y}_k \rangle|}{||\boldsymbol{x}_j|| \cdot ||\boldsymbol{y}_k||} = \frac{1}{\sqrt{d}} \quad \text{for all } j, k. \tag{1}$$

A set of orthogonal bases is a set of *mutually unbiased bases* (MUBs) if all pairs of distinct bases are unbiased.

The number of MUBs in $\mathbb{C}^d$ is at most $d+1$ [4, Table I], which can be attained when d is a prime power by a variety of methods [5,7,10]. The number of MUBs in $\mathbb{R}^d$ is at most $d/2+1$ [4, Table I], which can be attained when d is a power of 4 [1,2].

The authors recently gave a direct construction of $d^2/4$ equiangular lines in $\mathbb{C}^d$, where $d/2$ is a prime power [6]. We show here how to generalize the underlying construction to give $\Theta(d^2)$ equiangular lines in $\mathbb{C}^d$ and $\mathbb{R}^d$ directly from sets of complex and real MUBs.

2. The construction

We associate an ordered set of m vectors in $\mathbb{C}^d$ with the $m \times d$ matrix formed from the vector entries, using the ordering of the set to determine the ordering of the vectors.

Theorem 1. *Suppose that $B_1, B_2, \dots, B_r$ form a set of r MUBs in $\mathbb{C}^d$, each of whose vectors has all entries of unit magnitude, where $r \le d$. Let $a_1, a_2, \dots, a_t$ be constants in $\mathbb{C}$, where $t \ge 1$. Let $B_j(v)$ be the set of d vectors formed by multiplying entry j of each vector of B_j by $v \in \mathbb{C}$, and let $L(v) = \cup_{j=1}^r B_j(v)$ (considered as an ordered set). Then all inner products between distinct vectors among the rd vectors of*

$$\Big[L(a_1) \quad L(a_2) \quad \dots \quad L(a_t) \quad L\Big(t+1-\sum_{j=1}^t a_j\Big) \Big]$$

in $\mathbb{C}^{(t+1)d}$ have magnitude $\sum_{j=1}^t |a_j - 1|^2 + \Big|\sum_{j=1}^t (a_j - 1)\Big|^2$ or $(t+1)\sqrt{d}$.

Proof. Write $A = \{a_1, a_2, \dots, a_t, t+1-\sum_{j=1}^t a_j\}$ for the set of arguments $v \in \mathbb{C}$ taken by $L(v)$ in the construction. We consider two cases, according to whether distinct vectors of $L(v)$ originate from the same basis or from distinct bases.

In the first case, consider the inner product of distinct vectors of $L(v)$ constructed from vectors from the same basis B_j. Since the original vectors are orthogonal, this inner product is $z(|v|^2 - 1)$ for some z of unit magnitude that depends only on the original two