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Forbidding intersection patterns between layers of the cube

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A R T I C L E I N F O

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A family $\mathcal{A} \subset \mathcal{P}[n]$ is said to be an antichain if $A \not\subset B$ for all distinct $A, B \in \mathcal{A}$. A classic result of Sperner shows that such families satisfy $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$, which is easily seen to be best possible. One can view the antichain condition as a restriction on the intersection sizes between sets in different layers of $\mathcal{P}[n]$. More generally one can ask, given a collection of intersection restrictions between the layers, how large can families respecting these restrictions be? Answering a question of Kalai [8], we show that for most collections of such restrictions, layered families are asymptotically largest. This extends results of Leader and the author from [11].

1. Introduction

A family $\mathcal{A} \subset \mathcal{P}[n]$ is said to be an antichain if $A \not\subset B$ for all distinct $A, B \in \mathcal{A}$. A classic result in extremal combinatorics is Sperner's theorem [13], which shows that any such family $\mathcal{A}$ has size at most $\binom{n}{\lfloor n/2 \rfloor}$. This is easily seen to be best possible. This result has been hugely influential, having numerous interesting applications and extensions (for example, see [2] and [4] for an overview of some of these directions).

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Recently, Sperner's theorem was applied in a new proof of Furstenberg and Katznelson's density Hales–Jewett theorem by the polymath internet project ([12,6]). Here, roughly speaking, Sperner's theorem (and a multi-dimensional extension of Gunderson, Rödl and Sidorenko [7]) form a base level of an induction hypothesis. While weaker than Sperner's theorem, a crucial fact here was that any Sperner family $\mathcal{A} \subset \mathcal{P}[n]$ satisfies $|\mathcal{A}| = o(2^n)$.

Motivated by its place in the proof of the density Hales–Jewett theorem, Kalai [8] asked whether it is possible to obtain similar results for other 'Sperner-like conditions'. One example of such a condition was the tilted Sperner condition considered in [11]. Kalai noted that the Sperner condition can be rephrased as follows: $\mathcal{A}$ does not contain two sets A and B such that, in the unique subcube of $\mathcal{P}[n]$ spanned by A and B, A is the bottom point and B is the top point. He asked: what happens if we forbid A and B to be at a different position in this subcube? In particular, he asked how large $\mathcal{A} \subset \mathcal{P}[n]$ can be if we forbid A and B to be at a 'fixed ratio' $p : q$ in this subcube. That is, we forbid A to be $p/(p+q)$ of the way up this subcube and B to be $q/(p+q)$ of the way up this subcube. Equivalently, $q|A \setminus B| \neq p|B \setminus A|$ for all distinct $A, B \in \mathcal{A}$. Note that the Sperner condition corresponds to taking $p = 0$ and $q = 1$. In [11], an asymptotically tight answer was given for all ratios $p : q$, showing that one cannot improve on the 'obvious' example, namely the $q - p$ middle layers of $\mathcal{P}[n]$.

Theorem 1.1. *(See [11].) Let p, q be coprime natural numbers with $q \geq p$. Suppose $\mathcal{A} \subset \mathcal{P}[n]$ does not contain distinct A, B with $q|A \setminus B| = p|B \setminus A|$. Then*

$$|\mathcal{A}| \leq \big(q - p + o(1)\big)\binom{n}{n/2}. \tag{1.1}$$

Up to the $o(1)$ term, this is best possible. Indeed, the proof of Theorem 1.1 in [11] also gives the exact maximum size of such $\mathcal{A}$ for infinitely many values of n.

Here we will view the Sperner condition from a slightly different perspective. Given $i \in [0, n]$, let $[n]^{(i)} = \{A \subset \{1, \ldots, n\} : |A| = i\}$ and given a family of sets $\mathcal{A} \subset \mathcal{P}[n]$, let $\mathcal{A}^{(i)}$ denote the set $\mathcal{A}^{(i)} = \{A \in \mathcal{A} : |A| = i\}$.

Definition. A family $\mathcal{A} \subset \mathcal{P}[n]$ satisfies an x_{ij}*-pairwise restriction* between layers i and j of the cube if $|A \setminus B| \neq x_{ij}$ for all $A \in \mathcal{A}^{(i)}$ and $B \in \mathcal{A}^{(j)}$.

Both the Sperner and tilted Sperner conditions can be viewed as collections of pairwise restrictions between layers of the cube. Indeed, $\mathcal{A}$ is a Sperner family if and only if $|A \setminus B| \neq 0$ for all $A \in \mathcal{A}^{(i)}$ and $B \in \mathcal{A}^{(j)}$ whenever $i < j$. Similarly the tilted Sperner conditions can be viewed as collections of pairwise restrictions; for example, a small calculation shows that $\mathcal{A}$ is a $1 : 2$-tilted Sperner family if and only if $|A \setminus B| \neq j - i$ for all $A \in \mathcal{A}^{(i)}$ and $B \in \mathcal{A}^{(j)}$ for some pairs $\{i, j\}$ (those $i < j$ which satisfy $j \leq 2i$ and $2j - i \leq n$). The main question we consider in this paper is the following: given