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An extension of MacMahon's equidistribution theorem to ordered set partitions

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ABSTRACT

We prove a conjecture of Haglund which can be seen as an extension of the equidistribution of the inversion number and the major index over permutations to ordered set partitions. Haglund's conjecture implicitly defines two statistics on ordered set partitions and states that they are equidistributed. The implied inversion statistic is equivalent to a statistic on ordered set partitions studied by Steingrímsson, Ishikawa, Kasraoui, and Zeng and is known to have a nice distribution in terms of q-Stirling numbers. The resulting major index exhibits a combinatorial relationship between q-Stirling numbers and the Euler–Mahonian distribution on the symmetric group, solving a problem posed by Steingrímsson.

Contents

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1. Introduction

Let $\mathfrak{S}_n$ denote the symmetric group, i.e. the group of permutations of $\{1, 2, \dots, n\}$ under composition. Given a permutation $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$, we define the *descent* and *ascent sets* of σ to be

$$\begin{aligned}\operatorname{Des}(\sigma) &= \{i \in \{1, 2, \dots, n-1\} : \sigma_i > \sigma_{i+1}\} \quad \text{and} \\ \operatorname{Asc}(\sigma) &= \{i \in \{1, 2, \dots, n-1\} : \sigma_i < \sigma_{i+1}\}.\end{aligned}$$

The set of *inversions* of σ, $\operatorname{Inv}(\sigma)$, is defined by

$$\operatorname{Inv}(\sigma) = \{(i, j) : 1 \le i < j \le n,\ \sigma_i > \sigma_j\}.$$

Then

$$\operatorname{Inv}^{i,\square} = \{(i, j) : i < j \le n,\ \sigma_i > \sigma_j\}$$

is the set of inversions that start at position i and

$$\operatorname{Inv}^{\square,j} = \{(i, j) : 1 \le i < j,\ \sigma_i > \sigma_j\}$$

is the set of inversions that end at position j. We let

$$\begin{aligned}\operatorname{des}(\sigma) &= |\operatorname{Des}(\sigma)| & \operatorname{inv}(\sigma) &= |\operatorname{Inv}(\sigma)|, \\ \operatorname{asc}(\sigma) &= |\operatorname{Asc}(\sigma)| & \operatorname{inv}^{i,\square}(\sigma) &= |\operatorname{Inv}^{i,\square}(\sigma)|, \\ \operatorname{maj}(\sigma) &= \sum_{i \in \operatorname{Des}(\sigma)} i & \operatorname{inv}^{\square,j}(\sigma) &= |\operatorname{Inv}^{\square,j}(\sigma)|.\end{aligned}$$

The statistics $\operatorname{des}(\sigma)$, $\operatorname{asc}(\sigma)$, $\operatorname{maj}(\sigma)$, and $\operatorname{inv}(\sigma)$ are known as the *descent number*, *ascent number*, *major index*, and *inversion number* of σ, respectively.

This paper was motivated by the following conjecture of Jim Haglund (personal communication, October 2012).

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\operatorname{inv}(\sigma)} \prod_{i \in \operatorname{Des}(\sigma)} \left(1 + \frac{z}{q^{1+\operatorname{inv}^{\square,i}(\sigma)}}\right) = \sum_{\sigma \in \mathfrak{S}_n} q^{\operatorname{maj}(\sigma)} \prod_{j=1}^{\operatorname{des}(\sigma)} \left(1 + \frac{z}{q^j}\right). \tag{1}$$