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Characterizing partition functions of the edge-coloring model by rank growth

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A R T I C L E   I N F O

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A B S T R A C T

We characterize which graph invariants are partition functions of an edge-coloring model over $\mathbb{C}$, in terms of the rank growth of associated 'connection matrices'.

1. Introduction

Let $\mathcal{G}$ denote the collection of all undirected graphs, two of them being the same if they are isomorphic. In this paper, all graphs are finite and may have loops and multiple edges. Let $k \in \mathbb{N}$ and let $\mathbb{F}$ be a commutative ring. Call any function $y : \mathbb{N}^k \to \mathbb{F}$ a (k-*color*) *edge-coloring model* (*over* $\mathbb{F}$). In the case where y is symmetric under the action of S_k, an edge-coloring model is called a 'vertex model' by de la Harpe and Jones [5], where colors are called 'states'. The more general model was considered in the context of Holant functions by L.G. Valiant (cf. [1]).

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The *partition function* of an edge-coloring model y is the function $p_y : \mathcal{G} \to \mathbb{F}$ defined for any graph $G = (V, E)$ by

$$p_y(G) := \sum_{\kappa : E \to [k]} \prod_{v \in V} y_{\kappa(\delta(v))}.$$

Here $\delta(v)$ is the set of edges incident with v. Then $\kappa(\delta(v))$ is a multisubset of $[k]$, which we identify with its incidence vector in $\mathbb{N}^k$. Moreover, we use $\mathbb{N} = \{0, 1, 2, \ldots\}$ and for $n \in \mathbb{N}$,

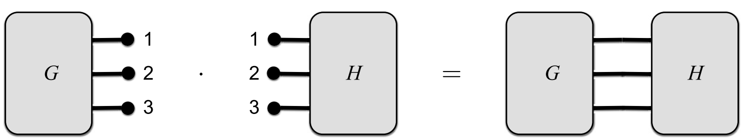
$$[n] := \{1, \ldots, n\}.$$

We can visualize κ as a coloring of the edges of G and $\kappa(\delta(v))$ as the multiset of colors 'seen' from v. The edge-coloring model was considered by de la Harpe and Jones [5] as a physical model, where vertices serve as particles, edges as interactions between particles, and colors as states or energy levels. It extends the Ising–Potts model. Several graph parameters are partition functions of some edge-coloring model, like the number of matchings. There exist real-valued graph parameters that are partition functions of an edge-coloring model over $\mathbb{C}$, but not over $\mathbb{R}$. (A simple example is $(-1)^{|E(G)|}$.)

In this paper, we characterize which functions $f : \mathcal{G} \to \mathbb{C}$ are the partition function of an edge-coloring model over $\mathbb{C}$. The characterization differs from an earlier characterization given in [3] (which our present characterization uses) in that it is based on the rank growth of associated 'connection matrices'.

To describe it, we need the notion of a k-fragment. For $k \in \mathbb{N}$, a *k-fragment* is an undirected graph $G = (V, E)$ together with an injective 'label' function $\lambda : [k] \to V$, where $\lambda(i)$ is a vertex of degree 1, for each $i \in [k]$. (You may alternatively view these degree-1 vertices as ends of 'half-edges', or rather of 'edge pieces', as both ends of an edge might be labeled.)

If G and H are k-fragments, the graph $G \cdot H$ is obtained from the disjoint union of G and H by identifying equally labeled vertices and by ignoring each of the k identified points as vertex, joining its two incident edges into one edge.



The multiplication $G \cdot H$.

(A good way to imagine this is to see a graph as a topological 1-complex.) Note that it requires (as in [8]) that we also should consider the 'vertexless loop' as possible edge of a graph, as we may create it in $G \cdot H$. We denote this vertexless loop by $\bigcirc$. Observe that if y is an edge-coloring model over $\mathbb{C}$ with n colors, then $p_y(\bigcirc) = n$.