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Almost-Fisher families

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ABSTRACT

A classic theorem in combinatorial design theory is Fisher's inequality, which states that a family $\mathcal{F}$ of subsets of $[n]$ with all pairwise intersections of size λ can have at most n non-empty sets. One may weaken the condition by requiring that for every set in $\mathcal{F}$, all but at most k of its pairwise intersections have size λ. We call such families k-almost λ-Fisher. Vu was the first to study the maximum size of such families, proving that for $k = 1$ the largest family has $2n - 2$ sets, and characterising when equality is attained. We substantially refine his result, showing how the size of the maximum family depends on λ. In particular we prove that for small λ one essentially recovers Fisher's bound. We also solve the next open case of $k = 2$ and obtain the first non-trivial upper bound for general k.

1. Introduction

A λ-*Fisher family* is a family $\mathcal{F}$ of sets with $|F_1 \cap F_2| = \lambda$ for all distinct $F_1, F_2 \in \mathcal{F}$. A fundamental result in combinatorial design theory, Fisher's inequality shows that a λ-Fisher family of subsets of $[n]$ can contain at most n non-empty sets. This simple

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restriction thus severely constricts the size of the family. One might hope to find larger families by weakening the conditions somewhat, allowing a limited number of 'bad' intersections.

To this end, we define a *k-almost λ-Fisher family* as a family $\mathcal{F}$ of sets such that for every $F \in \mathcal{F}$, there are at most k other sets $F' \in \mathcal{F}$ with $|F \cap F'| \neq \lambda$. We may then extend Fisher's inequality by determining how large a k-almost λ-Fisher family over $[n]$ can be. We denote this maximum by $f(n, k, \lambda)$. This problem was first introduced by Vu in 1999, who solved the case $k = 1$. In this paper we sharpen Vu's result, determining how the size of the maximum family depends on λ when $k = 1$, and solve the next open case of $k = 2$. We also obtain the first non-trivial upper bound for general k and provide a tight estimate for $\lambda = 0$.

We now discuss the background of this problem in greater detail before presenting our new results.

1.1. Restricted intersections

Extremal set theory is a rapidly developing area of combinatorics and has enjoyed tremendous growth in the past few decades. No doubt this is fuelled by its deep connections to other areas; extremal set theory both employs methods from and enjoys applications to diverse fields such as algebra, geometry and coding theory.

Many problems in extremal set theory are concerned with the pairwise intersections between sets in a family. For instance, much research concerns intersecting families, where empty pairwise intersections are forbidden. An intersecting family cannot contain a complementary pair of sets, and so can have size at most 2^{n-1}, a bound attained by many constructions. The celebrated Erdős–Ko–Rado theorem [5], one of the cornerstones of extremal set theory, provides the corresponding extremal result for k-uniform intersecting families.

Rather than forbidding empty intersections, we might seek to forbid other intersection sizes instead. Note that we would expect two uniformly random subsets of $[n]$ to intersect in $n/4$ elements. Erdős asked whether a family without a pairwise intersection of size exactly $n/4$ must have exponentially fewer than 2^n sets. This conjecture was resolved by Frankl and Rödl [7], who obtained a stronger result in the more general setting of codes (see [14] for a recent improvement).

As opposed to forbidding intersection sizes, one might instead require that all pairwise intersections be of the same size, and unsurprisingly this proves to be a much more restrictive condition. We call a family of sets a *λ-Fisher family* if any two distinct sets intersect in λ elements. The foundational result in this direction is Fisher's inequality [6], bounding the size of a λ-Fisher family. Fisher's original result dealt with more restrictive designs, and was extended to uniform λ-Fisher families by Bose [3]. The following non-uniform version was proven by Majumdar [16] and rediscovered by Isbell [13].

Theorem 1.1. *A λ-Fisher family over $[n]$ can have at most n non-empty sets.*