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Equiangular lines in Euclidean spaces

Gary Greaves [a], Jacobus H. Koolen [b], Akihiro Munemasa [a], Ferenc Szöllősi [a]

[a] *Research Center for Pure and Applied Mathematics, Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan*
[b] *School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui, 230026, PR China*

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ABSTRACT

We obtain several new results contributing to the theory of real equiangular line systems. Among other things, we present a new general lower bound on the maximum number of equiangular lines in d dimensional Euclidean space; we describe the two-graphs on 12 vertices; and we investigate Seidel matrices with exactly three distinct eigenvalues. As a result, we improve on two long-standing upper bounds regarding the maximum number of equiangular lines in dimensions $d = 14$ and $d = 16$. Additionally, we prove the nonexistence of certain regular graphs with four eigenvalues, and correct some tables from the literature.

1. Introduction

Let $d \geq 1$ be an integer and let $\mathbb{R}^d$ denote the Euclidean d-dimensional space equipped with the usual inner product $\langle .,. \rangle$. A set of $n \geq 1$ lines, represented by the unit vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^d$, is called *equiangular* if there exists a constant $\alpha \geq 0$, such that

E-mail addresses: grwgrvs@gmail.com (G. Greaves), koolen@ustc.edu.cn (J.H. Koolen), munemasa@math.is.tohoku.ac.jp (A. Munemasa), szoferi@gmail.com (F. Szöllősi).

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Table 1
The maximum number of equiangular lines for $d \leq 41$.

d	2	3	4	5	6	7–13	14	15	16	17	18	19	20	21	22	23–41
$N(d)$	3	6	6	10	16	28	28–29	36	40–41	48–50	48–61	72–76	90–96	126	176	276
$1/\alpha$	2	$\sqrt{5}$	$\sqrt{5}$, 3	3	3	3	3, 5	5	5	5	5	5	5	5	5	5

$\langle v_i, v_j \rangle = \pm\alpha$ for all $1 \leq i < j \leq n$. This constant is referred to as the *common angle* between the lines. If $\alpha = 0$ then the line system is just a subset of an orthonormal basis. If $n \leq d$ then it is easy to construct equiangular line systems for all $0 \leq \alpha \leq 1$. Hence we exclude these trivial cases by assuming that $n > d$ and consequently $\alpha > 0$.

Equiangular lines were introduced first by Haantjes [19] in 1948 and then investigated by Van Lint and Seidel in a seminal paper [29] and were further studied during the 1970s [6,9,36,47]. Recently there has been an interest in some special, highly structured *complex* equiangular line systems [17]. In the engineering literature these objects are called *tight frames*, and they are used for various applications in signal processing [15,22]. Some of these complex tight frames are studied by physicists under the name of *SIC-POVMs*, and are used in quantum tomography [1,39]. Discussion of the complex case and these applications are beyond the scope of this paper.

This paper is motivated by the fundamental problem of determining the *maximum number*, $N(d)$, *of equiangular lines* in $\mathbb{R}^d$. In Table 1 we display the current best known lower and upper bounds on $N(d)$ for the first few values of d. The table includes the contributions of this paper; compare to the relevant table in [26].

We remark that there exist several incorrectly revised tables in the current literature (see Remark 1.1), which might suggest to the uninitiated that $N(d)$ is known for small d. Table 1 shows that, despite a considerable amount of research in the past 40 years, determining $N(d)$ even for relatively small values of d is still out of reach. The methods used to obtain configurations with the above indicated number of lines are discussed throughout the scattered literature [26,41,48,47,49], while the upper bounds will be mentioned later. The reader may wish to jump ahead to Section 5.3, where the new results regarding dimensions $d = 14$ and $d = 16$ are presented.

Remark 1.1. Seidel seems to claim in [41, Section 3.3] that the lower bounds indicated in Table 1 above cannot be improved unless $d = 19$ or 20. This might be true, but it is unclear whether or not his statement follows implicitly from the cited literature.

The Gram matrix of the equiangular line system $[G]_{i,j} := \langle v_i, v_j \rangle$, $1 \leq i, j \leq n$, is of fundamental interest. It contains all the relevant parameters of the line system and thus the study of equiangular lines using matrix theoretical and linear algebraic tools is possible. We find it, however, more convenient to consider the *Seidel matrix* $S := (G - I)/\alpha$ instead, which is a symmetric matrix with zero diagonal and ± 1 entries otherwise. The multiplicity of the smallest eigenvalue $\lambda_0 = -1/\alpha$ of S describes the