Journal of Combinatorial Theory, Series A 132 (2015) 32–57

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Combinatorics of labelled parallelogram polyominoes ☆

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ARTICLE INFO

Article history:
Received 15 May 2013
Available online 19 December 2014

Keywords:
Parallelogram polyominoes
Representation theory
Macdonald polynomials

ABSTRACT

We obtain explicit formulas for the enumeration of labelled parallelogram polyominoes. These are the polyominoes that are bounded, above and below, by north-east lattice paths going from the origin to a point (k, n). The numbers from 1 to n (the labels) are bijectively attached to the n north steps of the above-bounding path, with the condition that they appear in increasing values along consecutive north steps. We calculate the Frobenius characteristic of the action of the symmetric group $\mathbb{S}_n$ on these labels. All these enumeration results are refined to take into account the area of these polyominoes. We make a connection between our enumeration results and the theory of operators for which the integral Macdonald polynomials are joint eigenfunctions. We also explain how these same polyominoes can be used to explicitly construct a linear basis of a ring of SL_2-invariants.

☆ This work has been supported by ANR-France (PSYCO project ANR-11-JS02-001), by CNRS-France (LIA LIRCO), by NSERC-Canada and by NSF-USA (NSF DMS10-68883).

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http://dx.doi.org/10.1016/j.jcta.2014.12.003
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1. Introduction

Parallelogram polyominoes have been studied by many authors (see [5,11,8] for a nice survey and enumeration results). They correspond to pairs $\pi = (\alpha, \beta)$ of north-east paths going from the origin to a point (k, n) in the combinatorial plane $\mathbb{N} \times \mathbb{N}$, with the path α staying "above" the path β. Our aim here is to study properties, and related nice formulas, of "labelled parallelogram polyominoes". These are obtained by bijectively labelling each of the n north steps of the path α with the numbers between 1 and n. Our motivation stems from a similarity between this new notion and recent work on labelled intervals in the Tamari lattice, in connection with the study of trivariate diagonal harmonic polynomials for the symmetric group (see [4]).

We calculate explicitly the Frobenius characteristic of the natural action of the symmetric group $\mathbb{S}_n$ on these labelled polyominoes; and study aspects of a weighted version of this Frobenius characteristic with respect to the area of the polyominoes. This connects our study to interesting operators for which adequately normalized Macdonald polynomials are joint eigenfunctions. This is the same theory that appears in the study of the $\mathbb{S}_n$-module of bi-variate diagonal harmonics (see [13,14,10]).

We also extend some of our considerations to parallelogram polyominoes, with added labels on east steps of the below-bounding path; with a corresponding action of the group $\mathbb{S}_k \times \mathbb{S}_n$. Several components of these spaces are naturally related to parking function modules.

2. Paths and labelled paths

Let k and n be two positive integers and set $N = k + n$. A $k \times n$ north-east (lattice) path in $\mathbb{N} \times \mathbb{N}$ is a sequence $\alpha = (p_0, \ldots, p_i, \ldots p_N)$ of points $p_i = (x_i, y_i)$ in $\mathbb{N} \times \mathbb{N}$, with $p_0 = (0, 0)$ and $p_N = (k, n)$, and such that

$$(x_{i+1}, y_{i+1}) = \begin{cases} (x_i, y_i) + (1, 0) & \text{an } \textit{east step}, \text{ or} \\ (x_i, y_i) + (0, 1) & \text{a } \textit{north step}. \end{cases}$$