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Extensions of Sperner and Tucker's lemma for manifolds ☆

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ABSTRACT

The Sperner and Tucker lemmas are combinatorial analogs of the Brouwer and Borsuk–Ulam theorems with many useful applications. These classic lemmas are concerning labellings of triangulated discs and spheres. In this paper we show that discs and spheres can be substituted by large classes of manifolds with or without boundary.

1. Introduction

1.1. Notation

Throughout this paper the symbol $\mathbb{R}^d$ denotes the Euclidean space of dimension d. We denote by $\mathbb{B}^d$ the d-dimensional ball and by $\mathbb{S}^d$ the d-dimensional sphere. If we consider $\mathbb{S}^d$ as the set of unit vectors x in $\mathbb{R}^{d+1}$, then points x and $-x$ are called *antipodal* and the symmetry given by the mapping $x \to -x$ is called the *antipodality* on $\mathbb{S}^d$.

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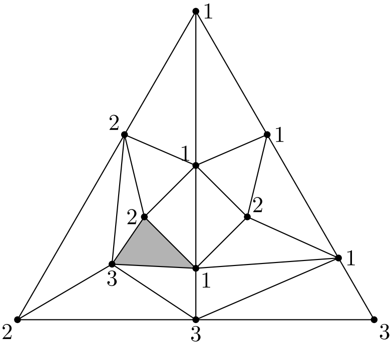

Fig. 1. A 2-dimensional illustration of Sperner's lemma.

1.2. Sperner's lemma

Sperner's lemma is a statement about labellings of triangulated simplices (d-balls). It is a discrete analog of the Brouwer fixed point theorem (see Fig. 1).

Let S be a d-dimensional simplex with vertices $v_1, \ldots, v_{d+1}$. Let T be a triangulation of S. Suppose that each vertex of T is assigned a unique label from the set $\{1, 2, \ldots, d+1\}$. A labelling L is called *Sperner's* if the vertices are labelled in such a way that a vertex of T belonging to the interior of a face F of S can only be labelled by k if v_k is on S.

Theorem 1.1 *(Sperner's lemma [13]). Every Sperner labelling of a triangulation of a d-dimensional simplex contains a cell labelled with a complete set of labels: $\{1, 2, \ldots, d+1\}$.*

There are several extensions of this lemma. One of the most interesting is the De Loera–Peterson–Su theorem. In the paper [4] they proved the Atanassov conjecture [1].

Theorem 1.2 *(Polytopal Sperner's lemma). Let P be a polytope in $\mathbb{R}^d$ with vertices $v_1, \ldots, v_n$. Let T be a triangulation of P. Let $L : V(T) \to \{1, 2, \ldots, n\}$ be a Sperner labelling. Then there are at least $(n-d)$ fully-colored (i.e. with distinct labels) d-simplices of T.*

Meunier [8] extended this theorem:

Theorem 1.3. *Let P^d be a d-dimensional PL manifold embedded in $\mathbb{R}^d$ that has boundary B. Suppose B has n vertices $v_1, \ldots, v_n$. Let T be a triangulation of P. Let $L : V(T) \to \{1, 2, \ldots, n\}$ be a Sperner labelling. Let d_i denote the number of edges of B which are connected to v_i. Then there are at least $n + \lceil \min_i\{d_i\}/d \rceil - d - 1$ fully-labelled d-simplices such that any pair of these fully-labelled simplices receives two different labellings.*